Sign test (符號檢定)

- Suppose that $(X_1, \ldots, X_n)$ is a random sample from $Bin(1, p)$. We have learned how to test

$$H_0 : p = p_0 \text{ versus } H_1 : p \neq p_0 \tag{1}$$

and

$$H_0 : p \leq p_0 \text{ versus } H_1 : p > p_0 \tag{2}$$

based on

$$Z_n = \frac{\sqrt{n}(\bar{X} - p_0)}{\sqrt{p_0(1 - p_0)}}.$$

The sign test is for testing problems that are special cases of (1) and (2), and the test is still based on $n\bar{X}$.

- The sign test can be used to test for difference for n paired observations. Below we will use new notation to describe the testing problem and the test statistc for the sign test.

- The sign test is based on results from n independent trials of two outcomes: "+" and "−". Let p_+ be the probability of getting a "+" sign in each trial. Two testing problems are of interest:

$$H_0 : p_+ = 0.5 \text{ versus } H_1 : p_+ \neq 0.5 \tag{3}$$

and

$$H_0 : p_+ \leq 0.5 \text{ versus } H_1 : p_+ > 0.5. \tag{4}$$

The sign test is based on two statistics: S^+ and S^-, where S^+ is the number of "+" outcomes and S^- be the number of "−" outcomes in the n trials.

- For the testing problem in (3), the sign test should reject $H_0 : p_+ = 0.5$ when S^+ is too large or $S^- = n - S^+$ is too large, which corresponds to the rejection rule

$$\max(S^+, S^-) \geq C_1. \tag{5}$$

Since $\max(S^+, S^-) = n - \min(S^+, S^-)$, (5) is equivalent to

$$\min(S^+, S^-) \leq n - C_1. \tag{6}$$

Here we use the rejection rule in (6) to make the evaluation of p-value more convenient. The test rejects $H_0 : p_+ = 0.5$ at level α if

$$\min(S^+, S^-) \leq C^*_{\alpha/2}$$

where $C^*_{\alpha/2}$ is the largest integer C such that

$$P(Bin(n, 0.5) \leq C) \leq \alpha/2.$$

The test rejects $H_0 : p_+ = 0.5$ at level α if and only if

$$\underbrace{2P\left(Bin(n, 0.5) \leq \text{ observed } \min(S^+, S^-)\right)}_{p\text{-value}} \leq \alpha.$$

- For the testing problem in (4), the sign test rejects $H_0 : p_+ \leq 0.5$ at level α if

$$S^- \leq C^*_\alpha,$$

where C^*_α is the largest integer C such that

$$P(Bin(n, 0.5) \leq C) \leq \alpha.$$

The sign test rejects the $H_0 : p_+ \leq 0.5$ at level α if and only if

$$\underbrace{P\left(Bin(n, 0.5) \leq \text{ observed } S^-\right)}_{p\text{-value}} \leq \alpha.$$

- Suppose that $(X_1, \ldots, X_n)$ is a random sample from a continuous distribution with a unique median μ. Consider the following testing problems:

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu \neq \mu_0, \tag{7}$$

$$H_0 : \mu \leq \mu_0 \text{ versus } H_1 : \mu > \mu_0 \tag{8}$$

and

$$H_0 : \mu \geq \mu_0 \text{ versus } H_1 : \mu < \mu_0. \tag{9}$$

- For testing (7) and (8), one can apply the sign test by treating $X_i > \mu_0$ as a "+" sign event since $\mu = \mu_0$ is equivalent to $p_+ = 0.5$ and $\mu \leq \mu_0$ is equivalent to $p_+ \leq 0.5$. Therefore, the statistics S^+ and S^- are given by

$$S^+ = \sum_{i=1}^{n} I(X_i > \mu_0) = \text{ number of } X_i\text{'s that are greater than } \mu_0,$$

and

$$S^- = \sum_{i=1}^{n} I(X_i < \mu_0) = \text{ number of } X_i\text{'s that are less than } \mu_0,$$

where

$$I(X_i > \mu_0) = \begin{cases} 1 & \text{if } X_i > \mu_0; \\ 0 & \text{otherwise.} \end{cases} \text{ and } I(X_i < \mu_0) = \begin{cases} 1 & \text{if } X_i < \mu_0; \\ 0 & \text{otherwise.} \end{cases}$$

- For testing (9), one can apply the sign test by treating $X_i < \mu_0$ as a "+" sign event since $\mu \geq \mu_0$ is equivalent to $p_+ \leq 0.5$.

- The R command `pbinom` can be used to compute $P(Bin(n,p) \leq C)$.
  - `pbinom((a:b), n, 0.5)` gives

$$P(Bin(n, 0.5) \leq a), \ldots, P(Bin(n, 0.5) \leq b).$$

- Example 1. Suppose that 5 people participated in a weight-losing program, and their weight losses are 0.9, 1.2, 1.9, 2.9, and 3.9 kilograms respectively. Suppose that the weight loss distribution is continuous and has a unique median. Based on the sign test, at level 0.2, can we conclude that the median of the weight loss distribution is

  (a) greater than 1 kilogram?

  (b) different from 1 kilogram?

  The R output after running the command

```
pbinom((0:5), 5, 0.5)
```

  is

```
[1] 0.03125 0.18750 0.50000 0.81250 0.96875 1.00000
```

  Sol.

  (a) When a participant's weight loss is greater than 1 kilogram, we treat it as a "+" sign event. Then, the alternative hypothesis that the median of the weight loss distribution is greater than 1 is equivalent to $p_+ < 0.5$, where p_+ is the probability that a "+" sign event occurs. The p-value for testing

$$H_0 : p_+ \leq 0.5 \text{ versus } H_1 : p_+ > 0.5$$

  using the sign test is

$$\begin{aligned} &P(Bin(5, 0.5) \leq \text{ observed } S^-) \\ &= P(Bin(5, 0.5) \leq 1) = 0.18750 < 0.2, \end{aligned}$$

  so at level 0.2, we can conclude that $p_+ > 0.5$. That is, at level 0.2, we can conclude that the median of the weight loss distribution is greater than 1 kilogram.

  (b) When a participant's weight loss is greater than 1 kilogram, we treat it as a "+" sign event. Then, the alternative hypothesis that the median of the weight loss distribution is different from 1 is equivalent to $p_+ \neq 0.5$, where p_+ is the probability that a "+" sign event occurs. The p-value for testing

$$H_0 : p_+ = 0.5 \text{ versus } H_1 : p_+ \neq 0.5$$

using the sign test is

$$\begin{aligned}
&2P(Bin(5,0.5) \leq \text{ observed } \min(S^+, S^-)) \\
&= 2P(Bin(5,0.5) \leq \min(4,1)) \\
&= 2P(Bin(5,0.5) \leq 1) = 0.18750 \times 2 > 0.2,
\end{aligned}$$

so we cannot conclude that $p_+ \neq 0.5$ at level 0.2. That is, we cannot conclude that the median of the weight loss distribution is different from 1 kilogram at level 0.2.

- Normal approximation to $Bin(n, 0.5)$. The normal approximation to $Bin(n, 0.5)$ with continuity correction is given below:

$$P(Bin(n, 0.5) \leq k) \approx P(N(n/2, n/4) \leq k + 0.5)$$

for $k \in \{0, \dots, n\}$.

- Example 2. 假設70名學生參加了語言培訓課程。他們在課程完成之前和完成之後參加了語言測驗，以檢查他們的語言技能是否有所提升。假設其中45名學生，培訓後的分數高於培訓前的分數，而另外25名學生，培訓前的分數高於培訓後的分數。 在0.05顯著水準下，是否可推論學生參加培訓後語言技能有提升? 使用有連續型校正的常態近似計算二項分布機率。

Sol. 定義"+"號事件爲一位學生培訓後的分數高於培訓前的分數，則 S^+ 爲培訓後的分數高於培訓前的分數之學生人數。令p_+爲"+"號事件發生機率， 則所考慮的檢定問題爲

$$H_0 : p_+ \leq 0.5 \text{ versus } H_1 : p_+ > 0.5.$$

針對上述檢定問題， sign test 的 p-value 爲

$$\begin{aligned}
P\left(Bin(70, 0.5) \leq \text{ observed } S^-\right) &= P\left(Bin(70, 0.5) \leq 25\right) \\
&\approx P(N(35, 17.5) < 25.5) \\
&= P(N(0,1) < (25.5 - 35)/\sqrt{17.5}) \\
&= P(N(0,1) > 9.5/\sqrt{17.5}) \\
&\approx P(N(0,1) > 2.27) \\
&= 0.5 - 0.4884 = 0.0116
\end{aligned}$$

因爲p-value 約爲$0.0116 < 0.05$, 在0.05顯著水準下，可推論學生參加培訓後語言技能有提升。