Covariance and correlation.

- Covariance. Suppose that X and Y are random variables with $E(X) = \mu_X$ and $E(Y) = \mu_Y$. Then the covariance of X and Y, denoted by $Cov(X, Y)$, is the expectation of $(X - \mu_X)(Y - \mu_Y)$. That is,

$$Cov(X, Y) = E((X - \mu_X)(Y - \mu_Y)) = E(XY) - E(X)E(Y). \quad (1)$$

  Note that when $Y = X$, $Cov(X, Y)$ is $Var(X)$ and (1) gives the formula $Var(X) = E(X^2) - (E(X))^2$.

- Suppose that X and Y are discrete random variables. Then $Cov(X, Y)$ can be determined if $P((X, Y) = (x, y))$ is given for each (x, y).

- Example 1. Suppose that

$$P((X,Y) = (x,y)) = \begin{cases} 0.1 & \text{if } (x,y) = (1,-2); \\ 0.3 & \text{if } (x,y) = (2,-4); \\ 0.6 & \text{if } (x,y) = (3,-6); \\ 0 & \text{otherwise.} \end{cases}$$

  Find $E(X)$, $E(Y)$, $E(XY)$ and $Cov(X, Y)$.

  Sol.

$$E(X) = 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.6 = 2.5,$$

$$E(Y) = (-2) \times 0.1 + (-4) \times 0.3 + (-6) \times 0.6 = -5,$$

$$E(XY) = (1)(-2) \times 0.1 + (2)(-4) \times 0.3 + (3)(-6) \times 0.6 = -13.4,$$

  and

$$Cov(X, Y) = E(XY) - E(X)E(Y) = -13.4 - (2.5) \times (-5) = -0.9.$$

- Rules for covariance calculation.

  1. $Cov(X, X) = Var(X)$.
  2. $Cov(X, Y) = Cov(Y, X)$.
  3. Linearity (in one variable when the other variable is held fixed).
     (i) For a constant a, $Cov(aX, Y) = aCov(X, Y) = Cov(X, aY)$.
     (ii) $Cov(X_1 + X_2, Y) = Cov(X_1, Y) + Cov(X_2, Y)$ and
     $Cov(X, Y_1 + Y_2) = Cov(X, Y_1) + Cov(X, Y_2)$.

  Note that the linearity of covariance implies that $Cov(aX, aX) = aCov(X, aX) = a^2Cov(X, X)$, which gives the formula $Var(aX) = a^2Var(X)$.

- Covariance under independence. If X and Y are independent, then $E(XY) = E(X)E(Y)$ and $Cov(X, Y) = 0$.
  - Example. $Cov(c, X) = 0$ for a constant c.
- From the rules for covariance calculation, we have

$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y) \tag{2}$$

  by expressing $Var(X+Y)$ as $Cov(X+Y, X+Y)$ and break it down using linearity of covariance. Another version of (2) is

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y)$$

  for constants a and b.
- Example 2. Suppose that $Cov(X, Y) = -0.9$, $Var(X) = 0.45$ and $Var(Y) = 1.8$. Find $Var(2X + Y)$.

  Sol.

$$\begin{aligned} Var(2X + Y) &= 2^2 \cdot Var(X) + 2 \cdot 2 \cdot 1 \cdot Cov(X, Y) + Var(Y) \\ &= 4 \times 0.45 + 4 \times (-0.9) + 1.8 = 0. \end{aligned}$$

- Correlation. Suppose that $Var(X) > 0$ and $Var(Y) > 0$. Then the correlation (or correlation coefficient) between X and Y, denoted by $Corr(X, Y)$, is

$$\frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}.$$

- Example 3. Consider the (X, Y) in Example 1. Find $Corr(X, Y)$.

  Sol. From the solution for Example 1, we have $Cov(X, Y) = -0.9$, $E(X) = 2.5$ and $E(Y) = -5$. Therefore,

$$Var(X) = E(X^2) - (2.5)^2 = 1^2 \times 0.1 + 2^2 \times 0.3 + 3^2 \times 0.6 - (2.5)^2 = 0.45,$$

$$Var(Y) = E(Y^2) - (-5)^2 = ((-2)\times 1)^2 \times 0.1 + ((-2)\times 2)^2 \times 0.3 + ((-2)\times 3)^2 \times 0.6 - (-5)^2 = 1.8,$$

  and

$$Corr(X, Y) = \frac{-0.9}{\sqrt{0.45 \times 1.8}} = -1.$$

- Properties of correlation. Suppose that $Var(X) > 0$ and $Var(Y) > 0$.
  1. $-1 \leq Corr(X, Y) \leq 1$.
  2. $|Corr(X, Y)| = 1 \Leftrightarrow Y = a + bX$ for some constants a and b.

3. Suppose that $Y = a + bX$ for some constants a and b (with probability one). Then

$$Corr(X, Y) = \begin{cases} 1 & \text{if } b > 0; \\ -1 & \text{if } b < 0. \end{cases}$$

4. If X and Y are independent, then $Corr(X, Y) = 0$.

5. Let $X^* = a + bX$ and $Y^* = c + dY$, where a, b, c, d are constants and $bd \neq 0$, then

$$Corr(X^*, Y^*) = \begin{cases} Corr(X, Y) & \text{if } bd > 0 \\ -Corr(X, Y) & \text{if } bd < 0. \end{cases}$$

- To see that $-1 \leq Corr(X, Y) \leq 1$, consider finding the minimum of $Var(Y + \lambda X)$, which is a quadratic function of λ. Solving

$$\frac{d}{d\lambda} Var(Y + \lambda X) = 0$$

gives $\lambda = -Cov(X, Y)/Var(X)$ and the minimum of $Var(Y + \lambda X)$ is

$$Var(Y + \lambda X)|_{\lambda = -Cov(X,Y)/Var(X)} = \frac{Var(X)Var(Y) - (Cov(X, Y))^2}{Var(X)}.$$

Since the minimum of $Var(Y + \lambda X)$ is nonnegative, we must have

$$Var(X)Var(Y) - (Cov(X, Y))^2 \geq 0,$$

which implies that $|Corr(X, Y)| \leq 1$.

- Note that

$$Corr(X, Y) = 0 \not\Rightarrow \ X \text{ and } Y \text{ are independent}$$

Example 4. Suppose that $P(X = 0) = P(X = 1) = P(X = -1) = 1/3$ and $Y = X^2$. Then $Cov(X, Y) = 0$ but X and Y are not independent since $P((X, Y) = (0, 0)) \neq P(X = 0)P(Y = 0)$.