Comparing two population variances

- F distribution. Suppose that $U_1 \sim \chi^2(d_1)$ and $U_2 \sim \chi^2(d_2)$ are independent. Then the distribution of

$$\frac{U_1/d_1}{U_2/d_2}$$

is called the F distribution with degrees of freedom (d_1, d_2), denoted by $F(d_1, d_2)$.

  - Note. Suppose $X \sim F(m,n)$, then $1/X \sim F(n,m)$.

- PDF of $F(m,n)$. Let Γ denote the gamma function, which is defined by

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

for $a > 0$. Define

$$f(x) = \frac{\Gamma((m+n)/2)}{\Gamma(m/2)\Gamma(n/2)} \left(\frac{m}{n}\right)^{m/2} x^{(m-2)/2} \left(1 + \frac{mx}{n}\right)^{-(m+n)/2} \tag{1}$$

for $x > 0$, and $f(x) = 0$ for $x \leq 0$. Then f is a PDF of $F(m,n)$.

- R commands related to the PDF of F distriubtions.

  - R command for computing $\Gamma(a)$ is `gamma(a)`.
  - R command for computing the $f(x)$ in (1) is `df(x,m,n)`.

- Plot the PDF of $F(30, 10)$ on $(0, 8)$ using R. R codes:

```
m <- 30
n <- 10
f <- function(x){
  ans <-  gamma((m+n)/2)/(gamma(m/2)*gamma(n/2))*(m/n)^(m/2)
  ans <- ans*x^((m-2)/2)
  ans <- ans*(1+m*x/n)^(-(m+n)/2)
  return(ans)
}
curve(f, 0,8)


g <- function(x){ df(x, 30, 10) }
curve(g, 0, 8, add=TRUE, col=2)
```

- Generate a random sample of size $n = 10^6$ from $F(30, 10)$ and compare the normalized histogram with the PDF of $F(30, 10)$ on the interval $(0, 8)$. R codes:

```
n <- 10^6
x <- rep(0, n)
d1 <- 30
d2 <- 10
for (i in 1:n){
 u1 <- sum(rnorm(d1)^2)
 #u1 is the sum of squares of d1 IID N(0,1) random variables
 u2 <- sum(rnorm(d2)^2)
 x[i] <- (u1/d1)/(u2/d2)
}
hist(x, nclass="scott", freq=FALSE, xlim =c(0,8))

g <- function(x){ df(x, 30, 10) }
curve(g, 0, 8, add=TRUE, col=2)
```

- Notation. $f_{a,m,n}$ denote the $(1-a)$ quantile of $F(m,n)$. That is,

$$P(F(d_1,d_2) > f_{a,d_1,d_2}) = a.$$

- Other R commands related to F distributions.
  - $f_{\mathtt{a,m,n}}$ can be computed by the R command `qf(1-a,m,n)`.
  - $P(F(\mathtt{m},\mathtt{n}) \leq \mathtt{C})$ can be computed by the R command `pf(C,m,n)`.

- Suppose that $(X_1,\ldots,X_{n_1})$ is a random sample from $N(\mu_1,\sigma_1^2)$ and $(Y_1,\ldots,Y_{n_2})$ is a random sample from $N(\mu_2,\sigma_2^2)$. Suppose that $(X_1,\ldots,X_{n_1})$ and $(Y_1,\ldots,Y_{n_2})$ are independent. Consider the testing problem

$$H_0 : \sigma_1 \geq \sigma_2 \text{ versus } H_1 : \sigma_1 < \sigma_2. \qquad (2)$$

- A reasonable statistic for testing (2) is S_Y^2/S_X^2, where S_X and S_Y are the sample standard deviations based on $(X_1,\ldots,X_{n_1})$ and $(Y_1,\ldots,Y_{n_2})$ respectively. When $\sigma_1 = \sigma_2$,

$$\frac{S_Y^2}{S_X^2} \sim F(n_2-1, n_1-1).$$

  The test statistic S_Y^2/S_X^2 is often called the F statistic and is denoted by F.

- For the testing problem in (2), the F test based on $F = S_Y^2/S_X^2$ rejects H_0 at level a when

$$F > f_{a,n_2-1,n_1-1}.$$

  The test is of size a. The p-value for the test is

$$P\left(F(n_2-1,n_1-1) > \text{ observed } F\right).$$

- Example 1. 假設有A和B二種降血壓藥物，考慮比較二藥效果。假設有二組病人，一組病人試用A藥而另一組病人試用B藥。

  - A 組人數: 10; 血壓降低量(收縮壓) sample mean: 15 (mm-Hg); sample standard deviation: 2.8 (mm-Hg).
  - B 組人數: 21; 血壓降低量 sample mean: 21 (mm-Hg); sample standard deviation: 3.2 (mm-Hg).

  假設不同病人用藥後血壓降低量爲獨立，而且使用A和B藥物後，血壓降低量分布分別爲爲$N(\mu_1, \sigma_1^2)$和$N(\mu_2, \sigma_2^2)$. 在 0.05 顯著水準下，是否可推論使用B藥物後，血壓降低量分布的variance 較大 ($\sigma_2 > \sigma_1$)?

  Sol. 檢定問題爲(2), F 統計量觀察值爲

$$\frac{3.2^2}{2.8^2} = 1.306122.$$

  由課程網站上的表 "0.95 quantiles for F distributions", 可知 $f_{0.05,21-1,10-1} = f_{0.05,20,9} = 2.94 > 1.306122$, 故在 0.05 顯著水準下，無法推論使用$B$藥物後, 血壓降低量分布的variance 較大。

  - We can also use R command `1-pf(1.306122, 20, 9)` to obtain the p-value, which is $0.3515168 > 0.05$.

- Suppose that $(X_1, \ldots, X_{n_1})$ is a random sample from $N(\mu_1, \sigma_1^2)$ and $(Y_1, \ldots, Y_{n_2})$ is a random sample from $N(\mu_2, \sigma_2^2)$. Suppose that $(X_1, \ldots, X_{n_1})$ and $(Y_1, \ldots, Y_{n_2})$ are independent. Consider the testing problem

$$H_0 : \sigma_1 = \sigma_2 \text{ versus } H_1 : \sigma_1 \neq \sigma_2. \tag{3}$$

  Let S_X and S_Y be the sample standard deviations based on $(X_1, \ldots, X_{n_1})$ and $(Y_1, \ldots, Y_{n_2})$ respectively. For the testing problem in (3), the F test rejects H_0 at level a when

$$\frac{S_Y^2}{S_X^2} > f_{a/2,n_2-1,n_1-1} \text{ or } \frac{S_X^2}{S_Y^2} > f_{a/2,n_1-1,n_2-1}.$$

  Note

  - The F test is of size a.
  - The p-value for the F test is

$$2\min\left(P\left(F(n_2-1, n_1-1) > \text{ observed } \frac{S_Y^2}{S_X^2}\right), P\left(F(n_1-1, n_2-1) > \text{ observed } \frac{S_X^2}{S_Y^2}\right)\right).$$

- Example 2. Example 1 中, 在 0.1 顯著水準下，是否可推論使用A, B二組血壓降低量分布的variances 不同?

Sol. 由課程網站上的表 "0.95 quantiles for F distributions", 可知 $f_{0.05,20,9} = 2.94$, $f_{0.05,9,20} = 2.39$. Example 1 中已計算

$$\frac{3.2^2}{2.8^2} = 1.306122 < f_{0.05,20,9}.$$

另外，

$$f_{0.05,9,20} = 2.39 > \frac{2.8^2}{3.2^2},$$

故在 0.1 顯著水準下, 不能推論使用A, B二組血壓降低量分布的variances 不同。

用以下 R 指令

```
qf(0.95, 20,9); qf(0.95, 9,20)
```

亦可算出 $f_{0.05,20,9}$ 和$f_{0.05,9,20}$.

- Example 3. 假設有A和B二種降血壓藥物，而我們想知道二藥效果是否有差。假設有二組病人，一組病人試用A藥而另一組病人試用B藥。A藥組有 10 人，B藥組有 13人。 假設不同病人用藥後血壓降低量爲獨立，而且使用A和B藥物後，血壓降低量分布分別爲爲$N(\mu_1, \sigma_1^2)$和$N(\mu_2, \sigma_2^2)$. 試用結果，A藥組血壓降低量爲 $X_1, \ldots, X_{10}$, 而B藥組血壓降低量爲 $Y_1, \ldots, Y_{13}$. 將 $X_1, \ldots, X_{10}$ 和 $Y_1, \ldots, Y_{13}$ 分別存於 R 中二向量 x 和 y. 假設執行以下 R 指令

```
2*(1-pf(var(x)/var(y), 9, 12))
2*(1-pf(var(y)/var(x), 12, 9))
t.test(x,y)$p.value
t.test(x,y, var.equal=T)$p.value
```

執行結果如下 (">"開始爲指令列，下一列爲運算結果)。

```
> 2*(1-pf(var(x)/var(y), 9, 12))
[1] 0.2186866
> 2*(1-pf(var(y)/var(x), 12, 9))
[1] 1.781313
> t.test(x,y)$p.value
[1] 0.04440313
> t.test(x,y, var.equal=T)$p.value
[1] 0.03154748
```

試回答以下問題：

(a) 以F檢定檢測H_0: $\sigma_1 = \sigma_2$ v.s. H_1: $\sigma_1 \neq \sigma_2$, F檢定之p-value 爲何？

(b) 在沒有強烈證據證明$\sigma_1 \neq \sigma_2$的情況下，我們將假設$\sigma_1 = \sigma_2$, 否則將假設$\sigma_1 \neq \sigma_2$. 在0.04顯著水準下，是否能合理推論二藥效果是否有差($\mu_1 \neq \mu_2$)?

(a) 根據給定的R執行結果，F檢定的p-value 爲

$$\min(0.2186866, 1.781313) = 0.2186866.$$

(b) 使用 two-sample t test (pooled t test) 時要假設二藥效分布的 variances 相同，而使用 Welch two-sample t test 時不必假設二藥效分布的 variances 相同。檢定二藥效分布的 variances 是否相同時，F檢定的p-value 爲p-value $= 0.2186866$, 因此並無強烈證據顯示 variances 不同，所以假設$\sigma_1 = \sigma_2$ 並使用 pooled t test. 根據給定的R執行結果，pooled t test之p-value爲$0.03154748 < 0.04$, 故在0.04顯著水準下可推論二藥效果不同。