Exponential distributions (指數分布).

- Exponential distributions are often used as the distributions of waiting times. The distribution of X is exponential means that

$$P(X \le x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x \ge 0; \\ 0, & \text{if } x < 0, \end{cases}$$

  where $\lambda > 0$ is a constant. We denote the distribution of X as $Exp(\lambda)$ in this handout. Note:

  - $X \sim Exp(\lambda) \Leftrightarrow \lambda X \sim Exp(1)$.
  - In some other places, the notation $Exp(\lambda)$ may represent the distribution $Exp(1/\lambda)$ here.

- Possion processes. For $t \ge 0$, let $N(t)$ be the number of occurrences of a rare event in the time interval $[0, t]$. $\{N(t) : t \ge 0\}$ is called a Poisson process if the following properties hold.

  1. $N(0) = 0$.
  2. Event occurrences during disjoint time intervals are independent. This implies that $N(t)$ and $N(t+s) - N(t)$ are independent for $s > 0$ and $t \ge 0$.
  3. $N(t+s) - N(t) \sim N(s)$ for $s > 0$ and $t \ge 0$.
  4. The event is a rare event. In particular,

$$\lim_{t \to 0} \frac{P(N(t) = 1)}{t} > 0 \text{ and } \lim_{t \to 0} \frac{P(N(t) \ge 2)}{t} = 0.$$

  Properties 1–4 together implies that $N(t) \sim Poisson(\lambda t)$, where

$$\lambda = \lim_{t \to 0} \frac{P(N(t) = 1)}{t}.$$

  The constant λ can be interpreted as the expected number of occurrences during a unit time interval, so λ is called the rate parameter for the Poisson process.

- Suppose that $\{N(t) : t \ge 0\}$ is a Poisson process with rate parameter $\lambda > 0$. Let X_i be the time for the i-th occurrence for $i \ge 1$. Then X_1, $X_2 - X_1$, $X_3 - X_2$, $\ldots$, are IID. It can be shown that $\lambda X_1 \sim Exp(1)$ using

$$P(X_1 > t) = P(N(t) = 0).$$

- Fact 1 If $X \sim Exp(1)$, then X has a PDF f, where for $x \in (-\infty, \infty)$,

$$f(x) = \begin{cases} e^{-x}, & \text{if } x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

  Note that if f is a PDF of $Exp(1)$, we must have that for $x \in (-\infty, \infty)$,

$$\int_{-\infty}^{x} f(s)ds = P(Exp(1) \leq x) = \begin{cases} 1 - e^{-x}, & \text{if } x \geq 0; \\ 0, & \text{if } x < 0. \end{cases} \quad (1)$$

  We will check (1) using R for $x > 0$. In such case, (1) can be simplified to

$$\int_{0}^{x} e^{-s}ds = 1 - e^{-x} \text{ for } x > 0.$$

  We will define two functions $F1$ and $F2$ in R, where $F1(x) = \int_0^x e^{-s}ds$ and $F2(x) = 1 - e^{-x}$ for $x > 0$. Running the following R commands and we can see that the graphs of $F1$ and $F2$ are the same on $(0, 10)$.

```
f <- function(s){ exp(-s) }
F <- function(x){ integrate(f, 0, x)$value }
F1 <- Vectorize(F)
F2 <- function(x){ 1-exp(-x)}
curve(F1, 0, 10)
curve(F2, 0, 10, add=T, col=2)
```

  Note that $F1$ is defined by `F1 <- Vectorize(F)`, which means that $F1(t) = F(t)$ for a number t, and for a vector $(t_1, \ldots, t_n)$, $F1(t_1, \ldots, t_n)$ is defined as the vector $(F(t_1), \ldots, F(t_n))$.

- Suppose that $\lambda > 0$ and $\lambda X \sim Exp(1)$. Let

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0; \\ 0, & \text{otherwise.} \end{cases}$$

  Then for $x \in (-\infty, \infty)$,

$$\int_{-\infty}^{x} f(s)ds = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x > 0; \\ 0, & \text{otherwise} \end{cases} = P(X \leq x).$$

  Thus f is a PDF of X.

- If $X \sim Exp(1)$, then $E(X) = 1$ and $Var(X) = 1$. The results can be verified using R.

  - Compute $E(X) = \int_0^\infty xe^{-x}dx$. Running the R commands

```
f <- function(x){ exp(-x) }
g <- function(x){ x*f(x) }
integrate(g,0, Inf)$value
```

gives the output

```
[1] 1
```

so we have $E(X) = \int_0^\infty xe^{-x}dx = 1$.

- Compute $E(X^2) = \int_0^\infty x^2 e^{-x}dx$. Running the R commands

```
f <- function(x){ exp(-x) }
g <- function(x){ (x^2)*f(x) }
integrate(g,0, Inf)$value
```

gives the output

```
[1] 2
```

so we have $E(X^2) = \int_0^\infty x^2 e^{-x}dx = 2$, which implies that $Var(X) = E(X^2) - (E(X))^2 = 2 - 1^2 = 1$.

- If $X \sim Exp(\lambda)$, then $E(X) = 1/\lambda$ and $Var(X) = 1/\lambda^2$.
- Since sthe notation $Exp(\lambda)$ in some other places may represent the distribution $Exp(1/\lambda)$ here, it is recommended that when using the notation $Exp(\lambda)$, you also give the mean of the exponential distribution for the sake of clarity.
- A lack-of-memory property of an exponential distribution. Suppose that $\lambda X \sim Exp(1)$, where $\lambda > 0$. Then for $s > 0$, $t > 0$,

$$P(X > s + t|X > s) = P(X > t).$$

- Example 1. Suppose that Ms. Yu goes fishing every weekend, and she catches 3 fishes in 2 hours on average. Suppose that for every $t > 0$, the number of fishes that Ms. Yu can catch during in the first t hours is a Poisson process. Let X be the waiting time (in hours) for Ms. Yu to catch the first fish. What is the distribution of X?

  Sol. Let $N(t)$ be the number of fishes caught in the first t hours. Let λ be the rate parameter for the Poisson process $\{N(t) : t \geq 0\}$, then $\lambda = 3/2 = 1.5$ and the distribution of X is $Exp(1.5)$, the exponential distribution with mean $1/1.5$.

- Example 2. In Example 1, find the probability that Ms. Yu does not catch any fish in the first half hour. Note that running the R command

```
exp(c(-0.25, -0.5, -0.75, -1))
```

gives

```
[1] 0.7788008 0.6065307 0.4723666 0.3678794
```

You may use the above R output for probability evaluation.
Sol. $P(X > 0.5) = 1 - (1 - e^{-1.5\times0.5}) = e^{-0.75} = 0.4723666$.