The cumulative distribution function (CDF) of a random variable

- For a random variable X, define a function F_X as follows:

$$F_X(x) = P(X \leq x) \text{ for } x \in (-\infty, \infty).$$

Then F_X is called the CDF (cumulative distribution function; 累積分布函數) of X.

- Example 1. Suppose that X has PMF p_X, where $p_X(0) = 0.6$ and $p_X(1) = 0.4$. Find the CDF of X.

Sol. Let F_X be the CDF of X. Then for $x < 0$,

$$F_X(x) = P(X \leq x) = 0.$$

For $0 \leq x < 1$,

$$F_X(x) = P(X \leq x) = P(X = 0) = p_X(0) = 0.6.$$

For $x \geq 1$,

$$F_X(x) = P(X \leq x) = P(X = 0) + P(X = 1) = p_X(0) + p_X(1) = 1.$$

In summary,

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0; \\ 0.6 & \text{if } 0 \leq x < 1; \\ 1 & \text{if } x \geq 1. \end{cases}$$

- Let F be the CDF of a random variable X, then

$$P(X = a) = \lim_{x \to a^+} F(x) - \lim_{x \to a^-} F(x) = F(a) - \lim_{x \to a^-} F(x).$$

That is, the jump amount of the graph of F at a is $P(X = a)$.

- Let F be the CDF of a random variable X, then for $b > a$,

$$F(b) - F(a) = P(a < X \leq b).$$

Example 2. Suppose that X is a random variable with CDF F, where

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ x & \text{if } 0 \leq x \leq 1; \\ 1 & \text{if } x > 1. \end{cases}$$

Find $P(0.5 < X \leq 2)$ and $P(0.5 < X < 2)$.

Sol.

$$P(0.5 < X \leq 2) = F(2) - F(0.5) = 1 - 0.5 = 0.5.$$

To find $P(0.5 < X < 2)$, note that

$$P(X = 2) = F(2) - \lim_{x \to 2^-} F(x) = 1 - 1 = 0,$$

so

$$\begin{aligned} P(0.5 < X < 2) &= P(0.5 < X \leq 2) - P(X = 2) \\ &= 0.5 - 0 = 0.5. \end{aligned}$$

- The distribution of a random variable can be characterized using its CDF. That is, for two random variables X and Y,

$$X \sim Y \Leftrightarrow \text{ } X \text{ and } Y \text{ have the same CDF}$$

- A random variable is continuous if its CDF is a continuous function on $(-\infty, \infty)$.

- If a random variable has a PDF, then it is a continuous random variable.