Mean, variance and standard deviation

- Suppose that X is a random variable. Let $X_1, \ldots, X_n, \ldots$ be IID random variables such that X_i and X have the same distribution. Then the mean (expectation; expected value; 期望值) of X is

$$\lim_{n\to\infty} \frac{(X_1 + \cdots + X_n)}{n},$$

  if the limit exists. We use $E(X)$ or μ to denote the mean of X.

- Some properties of expectation.
  - Suppose that X and Y are two random variables, then

$$E(X+Y) = E(X) + E(Y).$$

  - Suppose that X is a random variable and k is a constant (k is not random), then $E(kX) = kE(X)$.
  - For a constant k, $E(k) = k$.

- Suppose that X is a discrete random variable with PMF p_X. Then

$$E(X) = \sum_x x p_X(x). \tag{1}$$

  Here the sum is taken over all possible values for the distribution (all x's such that $p_X(x) > 0$).

- Example 1. Suppose that X is a discrete random variable with PMF p_X, where

$$p_X(x) = \begin{cases} 0.2 & \text{if } x = 0; \\ 0.3 & \text{if } x = 1; \\ 0.5 & \text{if } x = 2; \\ 0 & \text{otherwise.} \end{cases}$$

  Find $E(X)$.

  Sol. $E(X) = 0 \times 0.2 + 1 \times 0.3 + 2 \times 0.5 = 1.3$.

- For a random variable X with mean μ, the variance of X is

$$E(X-\mu)^2.$$

  We use $Var(X)$ to denote the variance of X.

- Note: $Var(X) = E(X^2) - (E(X))^2$.

- To compute $Var(X)$ for a discrete random variable X, the following result is used:

Fact 1 Suppose that X is a discrete random variable with PMF p_X, then for a real-valued function g such that $g(X)$ is defined,

$$E(g(X)) = \sum_{x:p_X(x)>0} g(x)p_X(x).$$

- If X is discrete with PMF p_X and mean μ, then

$$Var(X) = \sum_x (x-\mu)^2 p_X(x) = \left(\sum_x x^2 p_X(x)\right) - \mu^2. \tag{2}$$

Example 2. Find $Var(X)$ for the X in Example 1.

Sol. $Var(X) = (0-1.3)^2 \times 0.2 + (1-1.3)^2 \times 0.3 + (2-1.3)^2 \times 0.5 = 0.61$. Or,

$$\begin{aligned} Var(X) &= E(X^2) - (1.3)^2 \\ &= 0^2 \times 0.2 + 1^2 \times 0.3 + 2^2 \times 0.5 - 1.69 = 0.61. \end{aligned}$$

- The standard deviation of X is $\sqrt{Var(X)}$. We use σ to denote the standard deviation of X, so $Var(X)$ is also denoted by σ^2.

- Chebyshev's theorem. Suppose that $\mu = E(X)$ and $\sigma = \sqrt{Var(X)}$, then for $k > 0$,

$$P(|X-\mu| \le k\sigma) \ge 1 - \frac{1}{k^2}. \tag{3}$$

(3) is called Chebyshev's inequality.

Example 3. Suppose that X is a random variable whose mean and standard deviation are 75 and 2 respectively. Find an interval A such that $P(X \in A) \ge 0.8$.

  - Sol. Solving $1 - 1/k^2 = 0.8$ gives $k = \sqrt{5}$. By Chebyshev's Theorem, $P(|X-75| \le \sqrt{5} \times 2) \ge 0.8$. Take A to be the interval $[75-2\sqrt{5}, 75+2\sqrt{5}] \approx [70.52786, 79.47214]$, then $P(X \in A) \ge 0.8$.

- Suppose that X and Y are two random variables. If X and Y are independent, then

$$E(XY) = E(X)E(Y) \tag{4}$$

and

$$Var(X+Y) = Var(X) + Var(Y). \tag{5}$$

Note.

  - (4) can be verified using the PMF's of X and Y if X and Y are discrete.

  - (5) follows from (4).

- Suppose that X_1, ..., X_n are IID and $E(X_1) = \mu$, $Var(X_1) = \sigma^2$. Let $\bar{X} = (\sum_{i=1}^n X_i)/n$, then

$$E(\bar{X}) = \mu \tag{6}$$

  and

$$Var(\bar{X}) = \frac{\sigma^2}{n}. \tag{7}$$

  Note.

  - (7) follows from (5).
  - From (6), (7) and Chebyshev's theorem, if we have a random sample $(X_1, \ldots, X_n)$, it is reasonable to estimate $\mu = E(X_1)$ using $\bar{X}$.