A survey of probability concepts

- Objective probability: classical and empirical
- Events are represented by sets.
  - $\{a\}$ represents the event of getting outcome a.
  - $A \cap B$: A and B both happen.
  - $A \cup B$: A happens or B happens.
  - Example: 丟骰子出現奇數點的事件爲 $A = \{1, 3, 5\}$, 出現大於3點的事件爲 $B = \{4, 5, 6\}$, 出現奇數點且大於3點的事件爲 $A \cap B = \{5\}$.
- Classical probability
  - Possible outcomes from an experiment: a_1, ..., a_N.
  - $P(\{a_1\}) = \cdots = P(\{a_N\}) = 1/N$.
  - Suppose that $A = \{b_1, \ldots, b_m\} \subset \{a_1, \ldots, a_N\}$, then
$$P(A) = \frac{m}{N}.$$
  - Example: 丟骰子出現點數爲 i 的機率爲 $1/6$ $(1 \leq i \leq 6)$. 丟骰子出現奇數點的機率爲 $3/6 = 1/2$.
- Empirical probability.
$$\text{Empirical probability of } A = \frac{\text{number of times that } A \text{ occurs}}{\text{number of trials}}$$
  - Law of large numbers. If trials are independent and the probability of obtaining a particular outcome is the same for every trial, then
$$\text{empirical probability of } A \approx P(A)$$
    if the number of trials is large.
- Subjective probability: expert opinion
- Example. In each of the following cases, indicate whether classical or empirical probability is used.
  - (a) 某航空公司過去5年中平均每飛1000次出現飛航意外1次. 估計下次飛行現飛航意外的機率爲1/1000.
  - (b) 5名社團幹部, 3男2女, 要互相推選產生一位發言人. 5人決定用抽籤方式選出發言人. 選出女性發言人的機率爲2/5.

  Ans. (a) empirical probability (b) classical probability.
- Rules for probability calculation.

- Total probability rule. Let S_1 be the set of all possible outcomes, then $P(S_1) = 1$.
- Countable additivity. Suppose A_1, A_2, ... are mutually exclusive events (互斥事件; $A_i \cap A_j = \emptyset$ for $i \neq j$), then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots. \tag{1}$$

  (1) is also written as

$$P(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k).$$

- More rules for probability calculation (derived directly from the total probability rule and countable additivity)
  - $P(\emptyset) = 0$.
  - Complement rule. $P(A) + P(A^c) = 1$, where A^c is the complement (補集) of A.
  - General rule of addition (for two events). $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- The conditional probability of A given B, denoted by $P(A|B)$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

- Contingency tables (列聯表).
  - Example. 假設某汽車經銷商針對1000名顧客進行調查, 詢問顧客最近一次買車的時間以及是否想要換掉該次買的車, 結果如下表:

	最近一次買車的時間(距今)				
	2年內	2–3 年	3–5 年	5–10年	10年以上
想換車人數	2	15	90	95	60
不想換車人數	98	80	95	65	400

  假設從1000名顧客中隨機選出一名, 令A_1表示選出的顧客想要換車的事件. 令B_4表示選出的顧客最近一次買車時間距今已有10年以上. 求$P(A_1|B_4)$ 和 $P(B_4|A_1)$.

- Bayes theorem (貝氏定理). Suppose that events A_1, ..., A_n are mutually exclusive and collectively exhaustive (they form a partition of S) and $P(A_i) > 0$ for $1 \leq i \leq n$. Then for an event B such that $P(B) > 0$,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + \cdots + P(A_n)P(B|A_n)}$$

  for $1 \leq i \leq n$.

- In the text, $P(A_i)$'s are called the prior probabilities and $P(A_i|B)$'s are called the posterior probabilities.
- 假設工廠總共使用A,B兩條生產線生產某項產品, 其中生產線A負責生產20%, 不良率 5%, 生產線B負責生產80%, 不良率 1%. 假設現在隨機抽檢時發現了這項產品的一個不良品, 但不知是由生產線A或生產線B所生產. 用貝氏定理計算不良品來自生產線A的機率爲

$$\frac{0.2 \times 0.05}{0.2 \times 0.05 + 0.8 \times 0.01} = \frac{5}{9} \approx 0.56.$$

- Counting principles: multiplication formula, permutation formula and combination formula.
- Multiplication formula. Suppose that a process can be completed in two steps, where there are m ways to complete the first step and n ways to complete the second step. Then there are $m \times n$ ways to complete the process.
  - 假設搭公車從A站到C站必須經B站轉車. 從A站到B站有2線公車可搭, 從B站到C站有3線公車可搭, 則從A站搭公車到C站有$2 \times 3 = 6$種方式.
- Permutation formula. The number of ways of choosing r objects out of n objects and then putting them in order is $P_r^n = \dfrac{n!}{(n-r)!}$, where $n!$ means the product of $1, \ldots, n$.
  - 社團選舉, 由5個候選人選出正副社長, 共有 $5!/(5-2)! = 20$ 種方式.
- Combination formula. The number of ways of choosing r objects out of n objects is $C_r^n = \dfrac{n!}{r!(n-r)!}$.
  - 5人分成2組, 其中一組3人另一組2人, 分法有 $5!/(3!2!) = 10$種.
- Notation.
  - P_r^n is also written as ${}_nP_r$.
  - C_r^n is also written as ${}_nC_r$ or $\left(\begin{array}{c} n \\ r \end{array}\right)$.