Homework Problems

Note.

- Always show your work in the solution for each homework problem to receive full points unless it is stated otherwise in the problem.

1. (4 pts) Suppose that $\mathcal{F}$ is a σ-field on a space Ω and $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{F}$. Show that
$$(\cup_{n=1}^{\infty} A_n)^c = \cap_{n=1}^{\infty}(A_n^c)$$
by verifying that
$$(\cup_{n=1}^{\infty} A_n)^c \subset \cap_{n=1}^{\infty}(A_n^c)$$
and
$$(\cup_{n=1}^{\infty} A_n)^c \supset \cap_{n=1}^{\infty}(A_n^c).$$
Do not apply De Morgan's law to solve this problem.

2. (8 pts) Suppose that $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 3, 5\}$ and $B = \{1, 3\}$. Let $\mathcal{C} = \{\emptyset, \Omega, A, B\}$. Find $\sigma(\mathcal{C})$, the σ-field on Ω that is generated by $\mathcal{C}$. List all sets that should be included in $\sigma(\mathcal{C})$. You do not have to verify that the collection of sets in your list is a σ-field, but be sure it is. There are eight sets in $\sigma(\mathcal{C})$.

3. (4 pts) Suppose that $\Omega = \{1, 2, 3, 4, 5, 6\}$. In each of Cases (a)–(d), determine whether the collection $\mathcal{C}$ is a σ-field on Ω. You may write down your answers directly without justification.
   (a) $\mathcal{C} = \{\emptyset, \Omega\}$.
   (b) $\mathcal{C} = \{\emptyset, \Omega, A, B\}$, where $A = \{1, 3\}$ and $B = \{2, 4, 5\}$.
   (c) $\mathcal{C} = \{\emptyset, \Omega, A, B\}$, where $A = \{1, 3\}$ and $B = \{2, 4, 5, 6\}$.
   (d) $\mathcal{C} = \{\emptyset, \Omega, A, B, D\}$, where $A = \{1, 3\}$, $B = \{1, 2, 3\}$ and $D = \{4, 5, 6\}$.

4. (6 pts) Suppose that $\mathcal{F}$ is a σ-field on a space Ω and P is a probability function on $\mathcal{F}$. Suppose that $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets in $\mathcal{F}$, i.e. $A_n \supset A_{n+1}$ for all $n \in \{1, 2, \ldots\}$. Show that
$$P(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n).$$
You may also use the result in Problem 6 and the result in Fact 1 given below in your proof without proving them.

   Fact 1 *Suppose that $\mathcal{F}$ is a σ-field on a space Ω and P is a probability function on $\mathcal{F}$. Suppose that $\{A_n\}_{n=1}^{\infty}$ is an increasing sequence of sets in $\mathcal{F}$, i.e. $A_n \subset A_{n+1}$ for all $n \in \{1, 2, \ldots\}$. Then,*
$$P(\cup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} P(A_n).$$

5. (2 pts) Suppose that P is a probability function on $\mathcal{B}(R)$ such that for $n \in \{1, 2, \ldots, \}$,
$$P\left(\left(-\frac{1}{n}, \frac{1}{n}\right)\right) = 0.4 + \frac{0.6}{n}.$$
Find $P(\{0\})$. You may use Fact 1 in Problem 4 directly without proving it.

6. (2 pts) Suppose that $\mathcal{F}$ is a σ-field on a space Ω. Suppose that A, $B \in \mathcal{F}$ and $A \subset B$. Show that $B^c \subset A^c$.

7. (8 pts) Suppose that $\mathcal{F}$ is a σ-field on $\Omega = \{1, 2, 3, 4, 5\}$. Let $A = \{1, 2, 4, 5\}$, $B = \{1, 2, 4\}$ and $C = \{1, 4, 5\}$. Suppose that A, B, C are in $\mathcal{F}$ and P is a probability function defined on $\mathcal{F}$.

   (a) (4 pts) Find disjoint sets D_1, D_2, D_3 in $\mathcal{F}$ such that each of A, B and C can be expressed as the union of some sets in $\{D_1, D_2, D_3\}$. Write down those expressions.

   (b) (4 pts) Explain why we cannot have $(P(A), P(B), P(C)) = (0.5, 0.3, 0.1)$.

8. (6 pts) Suppose that F is the CDF of a random variable X, and $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence of real numbers such that $\lim_{n\to\infty} a_n = \infty$. Show that

$$\lim_{n\to\infty} F(-a_n) = 0$$

   without using the fact that $\lim_{x\to-\infty} F(x) = 0$. Hint: make use of the continuity of a probability function.

   Remark. It can be shown that $\lim_{x\to-\infty} F(x)$ exists, so the result in Problem 8 implies that $\lim_{x\to-\infty} F(x)$ is 0.

9. (6 pts) Suppose that $\mathcal{F}$ is a σ-field on a space Ω and P is a probability function defined on $\mathcal{F}$. For a set $B \in \mathcal{F}$ such that $P(B) > 0$, define a function Q on $\mathcal{F}$ by

$$Q(A) = \frac{P(A \cap B)}{P(B)}$$

   for $A \in \mathcal{F}$. Verify that Q is a probability function on $\mathcal{F}$.

10. (4 pts) Suppose that $\mathcal{F}$ is a σ-field on a space Ω and P is a probability function defined on $\mathcal{F}$. Suppose that A, $B \in \mathcal{F}$, $P(B) > 0$ and

$$P(A|B^c) = P(A|B) = p_0,$$

   where p_0 is a constant in $(0, 1)$. Show that $P(B|A) = P(B)$.

11. (4 pts) For the F in each of the following cases, determine whether F is a CDF. You may write down your answers directly without justification.

   (a) $F(x) = x$ for $x \in R$.

   (b) For $x \in R$,

$$F(x) = \begin{cases} 0.5 + 0.5\sin(x) & \text{if } x \in [0, 2\pi); \\ 0.5e^{-x} & \text{if } x < 0; \\ 1 - 0.5e^{-(x-2\pi)} & \text{if } x \geq 2\pi. \end{cases}$$

   (c) For $x \in R$,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0; \\ 1 & \text{if } x > 0. \end{cases}$$

   (d) For $x \in R$,

$$F(x) = \begin{cases} 0 & \text{if } x < 0.5; \\ x & \text{if } 0.5 \leq x < 1; \\ 1 & \text{if } x \geq 1. \end{cases}$$

12. (8 pts) Suppose that F is the CDF of a random variable X, and for $x \in R$,

$$F(x) = \begin{cases} 0 & \text{if } x < 0.4; \\ x & \text{if } 0.4 < x < 1; \\ 1 & \text{if } x > 1. \end{cases}$$

(a) (4 pts) Find $F(0.4)$ and $F(1)$.

(b) (4 pts) Find $P(X \in [0.4, 1])$.

13. (8 pts) Consider the experiment of rolling a die twice. Suppose that in each of the two trials, for $k \in \{1, 2, 3, 4, 5, 6\}$, the probability of rolling a k is $1/6$. Let X be the number of times (in the two trials) where a point less than 3 is rolled.

(a) (4 pts) Find the PMF of X.

(b) (4 pts) Find the CDF of X.

14. (8 pts) Suppose that X is a discrete random variable with PMF p_X, where for $x \in R$,

$$p_X(x) = \begin{cases} 0.2 & \text{if } x = -1; \\ 0.4 & \text{if } x = 0; \\ c \cdot (0.5)^x & \text{if } x \in \{1, 2, 3 \ldots, \}; \\ 0 & \text{otherwise,} \end{cases}$$

where $c > 0$ is a constant.

(a) (4 pts) Find c.

(b) (4 pts) Find $P(X > 25)$.

15. (4 pts) Suppose that X is a discrete random variable with PMF p_X, where for $x \in R$,

$$p_X(x) = \begin{cases} 0.2 & \text{if } x = 1; \\ 0.4 & \text{if } x = 2; \\ 0.4 & \text{if } x = 3; \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = |2 - X|$. Find the PMF of Y.

16. (16 pts) For $x \in R$, let

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ x & \text{if } 0 \le x < 1; \\ 1 & \text{if } x \ge 1. \end{cases}$$

(a) (6 pts) Verify that F is a CDF.

(b) (4 pts) Suppose that U is a random variable with CDF F. Show that $P(U \in (0, 1)) = 1$.

(c) (6 pts) Suppose that U is a random variable with CDF F and G is a continuous CDF such that G is a one-to-one function defined on R. Let G^{-1} be the inverse function of G. Show that the CDF of $G^{-1}(U)$ is G.

Note that in Part (c), $G^{-1}(U)$ can be defined with probability one. To see this, note that it can be shown (but you do not have to show this) that

$$(0,1) = \{G(x) : x \in R\},$$

so the domain of G^{-1} is $(0,1)$. Since U takes values in $(0,1)$ with probability one (the result in Part (b)), $G^{-1}(U)$ can be defined with probability one.

17. (8 pts) For $a, b \in R$ such that $a < b$, define a function $f_{a,b}$ on R as follows: for $x \in R$,

$$f_{a,b}(x) = \begin{cases} 1/(b-a) & \text{if } x \in (a,b); \\ 0 & \text{if } x \notin (a,b). \end{cases}$$

   (a) (4 pts) Suppose that U is a random variable with PDF $f_{0,1}$. For $a < b$, let $Y = a + (b-a)U$. Find a PDF of Y.

   (b) (4 pts) Suppose that X is a random variable with PDF $f_{a,b}$, where $a < b$. Let $U = (X-a)/(b-a)$. Find a PDF of U.

Note.

Definition 1 *For a random variable X with PDF $f_{a,b}$, the distribution of X is called the uniform distribution on (a,b), denoted by $U(a,b)$.*

We will use the notation $X \sim U(a,b)$ to denote that the distribution of X is $U(a,b)$. Your answers in (a) and (b) should give the following results:

$$U \sim U(0,1) \Rightarrow a + (b-a)U \sim U(a,b)$$

and

$$X \sim U(a,b) \Rightarrow (X-a)/(b-a) \sim U(0,1).$$

18. (16 pts) Suppose that X is a random variable with PDF f_X, where for $x \in R$,

$$f_X(x) = \begin{cases} 2xe^{-x^2} & \text{if } x > 0; \\ 0 & \text{if } x \le 0. \end{cases}$$

   (a) (4 pts) Find $P(0 < X < 1)$.

   (b) (4 pts) Find the CDF of X.

   (c) (4 pts) Find a PDF of $Y = \sqrt{X}$.

   (d) (4 pts) Let F be the CDF of X found in Part (b). Find a PDF of $F(X)$.

19. (4 pts) In each of the following cases, F is the CDF of a random variable X. Determine whether the distribution of $F(X)$ is $U(0,1)$ in each case, where the definition of $U(0,1)$ is given in Definition 1 after Problem 17. You may write down your answers directly without justification.

   (a) For $x \in R$,

$$F(x) = \begin{cases} 0 & \text{if } x < 0.5; \\ x & \text{if } 0.5 \le x < 1; \\ 1 & \text{if } x \ge 1. \end{cases}$$

(b) For $x \in R$,

$$F(x) = \int_{-\infty}^{x} f(t)dt,$$

where

$$f(x) = \begin{cases} 3/2 & \text{if } x \in (0, 1/3) \cup (2/3, 1); \\ 0 & \text{otherwise.} \end{cases}$$

(c)

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ x/3 & \text{if } 0 \leq x < 3; \\ 1 & \text{if } x \geq 3. \end{cases}$$

(d) For $x \in R$,

$$F(x) = \begin{cases} 0 & \text{if } x < 0.5; \\ 0.5 & \text{if } 0.5 \leq x < 1; \\ 1 & \text{if } x \geq 1. \end{cases}$$

Notion. From now on, we will use the notation I_S to denote the indicator function on a set S. That is,

$$I_S(x) = \begin{cases} 1 \text{ if } x \in S; \\ 0 & \text{if } x \notin S. \end{cases}$$

20. (6 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = |x| I_{(-1,0)}(x) + 0.5e^{-x} I_{(0,\infty)}(x)$$

Find a PDF of $Y = X^2$.

21. (4 pts) Suppose that F is the CDF of a random variable X such that

$$F'(x) = \begin{cases} 0 & \text{if } x < 1 \text{ or } x > 4; \\ f_1(x) & \text{if } x \in (1, 2); \\ f_2(x) & \text{if } x \in (2, 4), \end{cases}$$

where f_1 is continuous and nonnegative on $[1, 2]$ and f_2 is continuous and nonnegative on $[2, 4]$. Let

$$f(x) = f_1(x) I_{(1,2)}(x) + f_2(x) I_{(2,4)}(x)$$

for $x \in R$. Determine whether the statements in (a)–(d) are true. Write down your final answers only.

(a) $P(X \in (1, 2)) = \int_1^2 f_1(x)dx$.

(b) $P(X < 1) = 0$.

(c) If F is continuous at each point in $\{1, 2, 4\}$, then f is a PDF of X.

(d) If F is not continuous at some point in R, then f is not a PDF of X.

22. (8 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = e^{-x} I_{(0,\infty)}(x) \text{ for } x \in R.$$

Suppose that $Y = g(X)$, where

$$g(x) = x \cdot I_{(-\infty,0.5]}(x) + 0.5 \cdot I_{(0.5,\infty)}(x)$$

for $x \in R$.

(a) (4 pts) Find the CDF of Y.

(b) (2 pts) Explain why Y does not have a PDF.

(c) (2 pts) Find the quantile of order $(1-e^{-0.5})$ of the distribution of Y.

23. (20 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = c(xI_{(0,1]}(x) + 2I_{(1,2]}(x) + (3-x)I_{(2,3]}(x)) \text{ for } x \in R$$

and $c > 0$ is a constant.

(a) (2 pts) Find c.

(b) (4 pts) Find the median of the distribution of X.

(c) (6 pts) Find IQR of the distribution of X.

(d) (4 pts) Find $E(X)$.

(e) (4 pts) Find $E((X-1)^2)$.

24. (6 pts) Suppose that X is a random variable with PDF f_X, and

$$\{x : f_X(x) > 0\} = (a, b).$$

Suppose that g is continuous and strictly increasing on $[a, b]$, and $g' > 0$ on (a, b). Let $Y = g(X)$ and

$$f_Y(y) = f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| I_{(g(a),g(b))}(y)$$

for $y \in R$. Show that

$$\int_{-\infty}^{\infty} y f_Y(y)dy = \int_{-\infty}^{\infty} g(x)f_X(x)dx.$$

25. (8 pts) Suppose that X is a discrete random variable with PMF p_X, where for $x \in R$,

$$p_X(x) = \begin{cases} 0.2 & \text{if } x = 1; \\ 0.4 & \text{if } x = 2; \\ 0.4 & \text{if } x = 3; \\ 0 & \text{otherwise.} \end{cases}$$

(a) (4 pts) Find $E(X)$.

(b) (4 pts) Find $E(|2-X|)$.

26. (4 pts) Suppose that X has PDF f_X, where

$$f_X(x) = (2c_1/x^3)I_{(1,\infty)}(x) + (c_2/x^2)I_{(-\infty,-1)}(x)$$

for $x \in R$, and c_1, c_2 are nonnegative constants such that $c_1 + c_2 = 1$. For each of the following statements, determine whether the statement is true. You may write down your answers directly without justification.

(a) If $c_1 > 0$ and $c_2 > 0$, then $E(X) = -\infty$.

(b) If $c_1 > 0$ and $c_2 = 0$, then $E(X) = \infty$.

(c) If $c_1 > 0$ and $c_2 > 0$, the $E(|X|) = \infty$.

(d) If $c_1 > 0$ and $c_2 > 0$, the $E(X^3) = \infty$.

27. (8 pts) Suppose that $\mu > 0$ is a constant and X is a discrete random variable with PMF p_X, where for $x \in R$,

$$p_X(x) = \begin{cases} e^{-\mu}\mu^x/(x!) & \text{if } x \in \{0, 1, 2, \ldots, \} \\ 0 & \text{otherwise.} \end{cases}$$

   (a) (4 pts) Show that $E(X) = \mu$.

   (b) (4 pts) Show that $E(X(X-1)) = \mu^2$.

28. (6 pts) Suppose that m and M are two constants such that $m < M$, and X is a discrete random variable such that $P(X \in [m, M]) = 1$. Show that $m \leq E(X) \leq M$.

29. (8 pts) Suppose that X is a random variable such that both $E(X^2)$ and $E(|X|)$ are finite. Prove the following results.

   (a) (4 pts) $Var(X) = E(X^2) - (E(X))^2$;

   (b) (4 pts) $Var(X + c) = Var(X)$ for a constant c.

   You may apply Properties (i)–(iii) listed in Page 6 of the handout "Quantile and expectation" directly.

30. (8 pts) Suppose that X is a discrete random variable with PMF p_X, where

$$p_X(x) = \frac{e^{-\mu} \cdot \mu^x}{x!} I_{\{0,1,2,\ldots\}}(x) \tag{1}$$

   for $x \in R$, where $\mu > 0$ is a constant.

   (a) (4 pts) Find the MGF of X.

   (b) (4 pts) Show that $E(X) = \mu$ using the MGF of X.

   (c) (4 pts) Find $Var(X)$.

   Note. The distribution of X with the PMF p_X given in (1) is called the Poisson distribution with mean μ.

31. (8 pts) Suppose that X has PDF f, where

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

   for $x \in R$.

   (a) (4 pts) Show that

$$E(e^{tX}) = e^{t^2/2}$$

   for $t \in R$.

   (b) (2 pts) Show that $E(X) = 0$.

   (c) (2 pts) Show that $Var(X) = 1$.

   You may use the result in Part (a) to solve Part (b) and Part (c) without doing Part (a).

   Note. The distribution of X in Problem 31 is called the standard normal distribution, denoted by $N(0, 1)$.

32. (8 pts) Suppose that $X \sim N(0,1)$ (the distribution of X is $N(0,1)$). For $\mu \in R$ and $\sigma > 0$, let $Y = \mu + \sigma X$.

    (a) (4 pts) Find the MGF of Y.

    (b) (4 pts) Show that $E(Y) = \mu$ and $Var(Y) = \sigma^2$.

    Note. The distribution of Y in Problem 32 is called the normal distribution with mean μ and variance σ^2, denoted by $N(\mu, \sigma^2)$.

33. (6 pts) State and prove Chebyshev's inequality.

34. (2 pts) Suppose that X is a random variable with mean 1 and standard deviation 0.01. Find an interval (a,b) such that $b - a = 0.08$ and $P(X \in (a,b)) \geq 1 - (1/16)$.

35. (4 pts) Suppose that X and Y are random variables and F is the joint CDF of (X,Y). Show that for $(x_0, y_0) \in R^2$,

$$\lim_{n\to\infty} F\left(x_0, y_0 + \frac{1}{n}\right) = F(x_0, y_0)$$

    without using Properties (c)(f) in Page 1 of the handout "Joint distribution of a vector of random variables".

36. (8 pts) Suppose that X and Y are discrete random variables, and

$$P((X,Y) = (x,y)) = \begin{cases} 0.5 & \text{if } (x,y) = (1,2); \\ 0.1 & \text{if } (x,y) = (3,2); \\ 0.3 & \text{if } (x,y) = (3,6); \\ 0.1 & \text{if } (x,y) = (3,7); \\ 0 & \text{otherwise.} \end{cases}$$

    It can be shown that for discrete random variables X and Y,

$$E(g(X,Y)) = \sum_{(x,y):P((X,Y)=(x,y))>0} g(x,y)P((X,Y) = (x,y)) \qquad (2)$$

    if g is nonnegative.

    (a) (4 pts) Find $E(XY)$ using (2).

    (b) (4 pts) Find $E(XY)$ using the PMF of XY.

37. (8 pts) Suppose that (X,Y) is a random vector with CDF $F_{X,Y}$, where for $(x,y) \in R^2$,

$$F_{X,Y}(x,y) = 0.5G(x)G(y) + 0.5(1 - e^{-x})(1 - e^{-y})I_{[0,\infty)}(x)I_{[0,\infty)}(y),$$

    and the function G is defined by

$$G(x) = xI_{(0,1)}(x) + I_{[1,\infty)}(x)$$

    for $x \in R$.

    (a) (4 pts) Find $P(0 < X \leq 1 \text{ and } 1 < Y \leq 2)$.

    (b) (4 pts) Find the CDF of X.

    (c) (4 pts) Find $P(0 \leq X \leq 1 \text{ and } 1 < Y \leq 2)$.

38. (4 pts) Suppose that (X, Y, Z) is a random vector with joint CDF F. Show that

$$\begin{aligned} &P((X, Y, Z) \in (a, b] \times (c, d] \times (e, f]) \\ &= F(b, d, f) - F(b, c, f) - F(a, d, f) + F(a, c, f) \\ &-F(b, d, e) + F(b, c, e) + F(a, d, e) - F(a, c, e), \end{aligned}$$

where a, b, c, d, e, f are constants such that $a < b$, $c < d$ and $e < f$.

39. (8 pts) Suppose that (X, Y) has PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = c \cdot x I_{(0,1)}(x) I_{(0,1)}(y)$$

for $(x, y) \in R^2$ and $c > 0$ is a constant.

(a) (4 pts) Show that $c = 2$.

(b) (4 pts) Find a PDF of Y.

40. (4 pts) Suppose that (X, Y) has PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = 2x I_{(0,1)}(x) I_{(0,1)}(y)$$

for $(x, y) \in R^2$. Find $P(X + 2Y \leq 1)$.

41. (4 pts) Suppose that $(x_0, y_0) \in R^2$, $\delta > 0$, and u and v are two real-valued functions that are differentiable on

$$B((x_0, y_0), \delta) = \{(x, y) : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\}.$$

Let

$$J(x, y) = \begin{pmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{pmatrix}$$

for $(x, y) \in B((x_0, y_0), \delta)$. Suppose that the determinant of $J(x, y) \neq 0$ for $(x, y) \in B((x_0, y_0), \delta)$. For $h \in (0, \delta/\sqrt{2})$, let R_h be the region inside the parallelogram $ABCD$, where $A = (u_0, v_0)$, $B = (u_0 + u_x(x_0, y_0)h, v_0 + v_x(x_0, y_0)h)$,

$$C = (u_0 + u_x(x_0, y_0)h + u_y(x_0, y_0)h, v_0 + v_x(x_0, y_0)h + v_y(x_0, y_0)h)$$

and $D = (u_0 + u_y(x_0, y_0)h, v_0 + v_y(x_0, y_0)h)$. Show that

$$\lim_{h \to 0^+} \frac{\text{area of parallelogram } ABCD}{h^2} = |\text{determinant of } J(x_0, y_0)|\,.$$

42. (4 pts) Suppose that S is an open set in R^2 and u and v are two real-valued functions that are differentiable on S. Let $H(x, y) = (u(x, y), v(x, y))$ for $(x, y) \in S$. Suppose that H is a one-to-one function on S. Let $T = \{H(x, y) : (x, y) \in S\}$. Suppose that $H^{-1}(u, v) = (x(u, v), y(u, v))$ for $(u, v) \in T$ and $x(u, v)$ and $y(u, v)$ are functions of (u, v) that are differentiable on T. Show that

$$\begin{pmatrix} u_x(x(u, v), y(u, v)) & u_y(x(u, v), y(u, v)) \\ v_x(x(u, v), y(u, v)) & v_y(x(u, v), y(u, v)) \end{pmatrix} \begin{pmatrix} x_u(u, v) & x_v(u, v) \\ y_u(u, v) & y_v(u, v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for $(u, v) \in T$.

43. (10 pts) Suppose that (X, Y) has PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = ce^{-(x^2+y^2)/2} I_{(0,\infty)}(x) I_{(0,\infty)}(y)$$

for $(x, y) \in R^2$ and $c > 0$ is a constant. Let $U = \sqrt{X^2 + Y^2}$ and $V = \tan^{-1}(Y/X)$. Note that for $z \in (-\infty, \infty)$, $\tan^{-1}(z)$ is the value $\theta \in (-\pi/2, \pi/2)$ such that $\tan(\theta) = z$.

(a) (4 pts) Find a PDF of (U, V). Leave the constant c in your answer.

(b) (4 pts) Find a PDF of V. Leave the constant c in your answer.

(c) (2 pts) Find c using your answer in Part (b) and the fact that the integral of a PDF of V over $(-\infty, \infty)$ is 1.

Remark. The result from Problem 43(c) can be used for finding $\int_{-\infty}^{\infty} e^{-x^2/2} dx$. To see this, let $I = \int_0^{\infty} e^{-x^2/2} dx$, then

$$1 = \int_{R^2} f_{X,Y}(x, y) d(x, y) = cI^2,$$

so $I = 1/\sqrt{c}$ and $\int_{-\infty}^{\infty} e^{-x^2/2} dx = 2I = 2/\sqrt{c}$ (you should be able to obtain $2/\sqrt{c} = \sqrt{2\pi}$ if your answer for c is correct).

44. (10 pts) Let Γ be the function on $(0, \infty)$ defined by

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx$$

for $a > 0$. Suppose that α and β are two positive constants and (X, Y) has joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} e^{-x} I_{(0,\infty)}(x) y^{\beta-1} e^{-y} I_{(0,\infty)}(y)$$

for $(x, y) \in R^2$.

(a) (4 pts) Let M be the function on $(-\infty, 1) \times (-\infty, 1)$ defined by

$$M(t_1, t_2) = \left(\frac{1}{1-t_1}\right)^{\alpha} \left(\frac{1}{1-t_2}\right)^{\beta}$$

for $(t_1, t_2) \in (-\infty, 1) \times (-\infty, 1)$. Show that M is the joint MGF of (X, Y).

(b) (2 pts) Find the MGF of X.

(c) (4 pts) Find $E(XY)$ and $E(X)$.

45. (2 pts) Consider the (X, Y) in Problem 44. The distribution of X is called the gamma distribution with shape parameter α and scale parameter 1, denoted by $\Gamma(\alpha, 1)$. Show that the distribution of $(X + Y)$ is $\Gamma(\alpha + \beta, 1)$. Hint: the MGF of $(X+Y)$ can be easily obtained from the MGF of (X, Y).

46. (8 pts) Suppose that (X, Y) has a joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = cI_S(x, y)$$

for $(x, y) \in R^2$,

$$S = \{(x, y) : -2 < x + 2y < 2 \text{ and } -2 < x - 2y < 2\},$$

and $c = 1/\int_{R^2} I_S(x, y) d(x, y)$.

(a) (4 pts) Find $P((X + 2Y) > 0)$. You may leave c in your answer.

(b) (4 pts) Find c.