Homework Problems

- Note. Always show your work in your homework solutions to receive full points unless it is stated otherwise.

1. (6 pts) Suppose that $\mathcal{F}$ is a σ-field on a space Ω and $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{F}$. Show that

$$(\cup_{n=1}^{\infty} A_n)^c = \cap_{n=1}^{\infty}(A_n^c).$$

2. (6 pts) Suppose that P is a probability function defined on a σ-field $\mathcal{F}$. For a set $B \in \mathcal{F}$ such that $P(B) > 0$, define a function Q on $\mathcal{F}$ by

$$Q(A) = \frac{P(A \cap B)}{P(B)}$$

for $A \in \mathcal{F}$. Verify that Q is a probabilty function on $\mathcal{F}$.

3. (6 pts) Suppose that $\mathcal{F}$ is a σ-field on a space Ω and P is a probability function on $\mathcal{F}$. Suppose that $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{F}$ such that $A_n \supset A_{n+1}$ for all $n \in \{1, 2, \dots\}$. Show that

$$P\left(\cap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

You may use the following fact in your proof.

Fact 1 Suppose that $\mathcal{F}$ is a σ-field on a space Ω and P is a probability function on $\mathcal{F}$. Suppose that $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{F}$ such that $A_n \subset A_{n+1}$ for all $n \in \{1, 2, \dots\}$. Then,

$$P\left(\cup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

4. (4 pts) Suppose that $\mathcal{F}$ is a σ-field on $\Omega = (-\infty, \infty)$ such that all open intervals in $(-\infty, \infty)$ are in $\mathcal{F}$. Suppose that P is a probability function on $\mathcal{F}$ such that for $n \in \{1, 2, \dots, \}$,

$$P\left(\left(-\frac{1}{n}, \frac{1}{n}\right)\right) = 0.4 + \frac{0.6}{n}.$$

Find $P(\{0\})$.

5. (8 pts) Suppose that $\mathcal{F}$ is a σ-field on $\Omega = \{1, 2, 3, 4, 5\}$. Let $A = \{1, 2, 4, 5\}$, $B = \{1, 2, 4\}$ and $C = \{1, 4, 5\}$. Suppose that A, B, C are in $\mathcal{F}$ and P is a probability function defined on $\mathcal{F}$. Note that each of $P(A)$, $P(B)$ and $P(C)$ can be expressed as linear combinations of $P(\{5\})$, $P(\{1, 4\})$ and $P(\{2\})$. Use the linear relations to express $P(\{5\})$, $P(\{1, 4\})$, $P(\{2\})$ and $P(\{3\})$ in terms of $P(A)$, $P(B)$ and $P(C)$, and explain why we cannot have

$$(P(A), P(B), P(C)) = (0.5, 0.3, 0.1).$$

6. (4 pts) Suppose that A_1, A_2, A_3, A_4 are events in a σ-field on Ω. Suppose that $P(A_i) = 0.1$ and $P(A_i \cap A_j) = 0.05$ for $i,\ j \in \{1, 2, 3, 4\}$. Find a lower bound and an upper bound for $P(A_1 \cup A_2 \cup A_3 \cup A_4)$. Be sure that the lower bound is greater than 0 and the upper bound is less than 1.

7. (8 pts) Suppose that $\Omega = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Let $\mathcal{C} = \{\emptyset, \Omega, A, B\}$. Suppose that $\mathcal{F}$ is the smallest σ-field on Ω and $\mathcal{C} \subset \mathcal{F}$. List all sets that should be included in $\mathcal{F}$ and explain why they should be in $\mathcal{F}$. You do not have to verify that the collection of sets in your list is a σ-field, but be sure that it is.

8. (6 pts) Suppose that F is the CDF of a random variable X. Show that

$$\lim_{x \to a^+} F(x) = F(a)$$

for all $a \in R$. You may use the fact that $\lim_{x \to a^+} F(x)$ exists for all $a \in R$ without proving it.

Hint: find a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ so that $\cap_{n=1}^{\infty} A_n = (-\infty, a]$ and then use the continuity of a probability function to establish the result.

9. (6 pts) Suppose that X is a random variable with CDF F, where

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ 0.5 + 0.5x & \text{if } 0 \le x < 1; \\ 1 & \text{if } x \ge 1; \end{cases}$$

(a) (3 pts) Find $P(X = a)$ for every $a \in R$.

(b) (3 pts) Find $P(0 \le X \le 1)$.

10. (6 pts) For a, $b \in R$ such that $a < b$, define a function $f_{a,b}$ on R as follows: for $x \in R$,

$$f_{a,b}(x) = \begin{cases} 1/(b-a) & \text{if } x \in (a, b); \\ 0 & \text{if } x \notin (a, b). \end{cases}$$

Suppose that X is a random variable with PDF $f_{a,b}$. Show that $f_{0,1}$ is a PDF of $(X - a)/(b - a)$.

Notation.

- In Problem 10, the distribution of X is called the uniform distribution on (a, b), denoted by $U(a, b)$.
- We will write $X \sim U(a, b)$ to indicate the distribution of X is $U(a, b)$.
- We will use the notation I_A to denote the indicator function of A for a given set A, which is defined by

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$$

For instance, the $f_{a,b}(x)$ defined in Problem 10 is $I_{(a,b)}(x)$.

11. (6 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = 2xe^{-x^2} I_{(0,\infty)}(x) \text{ for } x \in R.$$

Find a PDF of $Y = \sqrt{X}$.

12. (12 pts) Consider the following function F:

$$F(x) = \begin{cases} c_1 & \text{if } x < 0; \\ 0.5 + 0.5x & \text{if } 0 < x < 1; \\ c_2 & \text{if } x > 1; \\ c_3 & \text{if } x = 0; \\ c_4 & \text{if } x = 1, \end{cases}$$

where c_1, c_2, c_3, c_4 are constants. Suppose that F is the CDF of some random variable X.

(a) (8 pts) Use the properties of a CDF given in Page 2 of the handout "Random variables" to find c_1, c_2, c_3, c_4. The link for the handout is

`https://stat.walkup.tw/teaching/math_stat_under/handouts/C01_5_random_variable.pdf`

(b) (4 pts) Explain why X is not a discrete random variable.

13. (6 pts) Suppose that X is a discrete random variable with PMF p_X, which is given below:

$$p_X(x) = \begin{cases} 0.2 & \text{if } x = -1; \\ 0.4 & \text{if } x = 0; \\ c \cdot (0.5)^x & \text{if } x \in \{1, 2, 3 \ldots, \}; \\ 0 & \text{otherwise,} \end{cases}$$

where $c > 0$ is a constant.

(a) (3 pts) Find c.

(b) (3 pts) Find $P(X > 25)$.

14. (6 pts) Suppose that X is a random variable with PDF f_X. Let $S_X = \{x : f_X(x) > 0\}$. Suppose that S_X is an open interval and f_X is continuous on S_X. Let F be the CDF of X, then it can be shown that $F' > 0$, F is continuous on S_X, and the inverse function F^{-1} is defined on $(0, 1)$ and differentiable on $(0, 1)$ (but you don't have to prove these results). Suppose that $U \sim U(0, 1)$, that is, the $f_{0,1}$ defined in Problem 10 is a PDF of U. Show that f_X is a PDF of $F^{-1}(U)$.

Remark. The result that X and $F^{-1}(U)$ have the same distribution can be established under weaker conditions.

15. (6 pts) Suppose that X is a random variable with CDF F, where

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ 1 - e^{-2x} & \text{if } x \geq 0. \end{cases}$$

(a) (2 pts) Find $P(X > 2)$.

(b) (4 pts) Find a PDF of X.

16. (6 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = |x| I_{(-1,0)}(x) + 0.5 e^{-x} I_{(0,\infty)}(x)$$

Find a PDF of $Y = X^2$.

Hint: find the CDF of Y first, take the derivative of the CDF as a guess of the PDF of Y, and then verify it is the PDF of Y.

17. (6 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = e^{-x} I_{(0,\infty)}(x) \text{ for } x \in R.$$

Suppose that

$$Y = \begin{cases} X & \text{if } X \leq 0.5; \\ 0.5 & \text{if } X > 0.5. \end{cases}$$

Find the CDF of Y and explain why Y does not have a PDF.

18. (8 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = 2xe^{-x^2} I_{(0,\infty)}(x) \text{ for } x \in R.$$

   (a) (4 pts) Find the CDF of X.

   (b) (4 pts) Find the median and the IQR of the distribution of X.

19. (4 pts) Suppose that X is a random variable such that both $E(X^2)$ and $E(|X|)$ are finite. Verify that $Var(X) = E(X^2) - (E(X))^2$ using Properties (i)–(iii) listed in Page 6 of the handout "Quantile and expectation".

20. (4 pts) Suppose that X is a random variable with finite expectation μ and standard deviation $\sigma > 0$. Let $Y = (X - \mu)/\sigma$. Find $E(Y)$ and $Var(Y)$ with $\mu = 1.5$ and $\sigma = 1.2$.

21. (4 pts) Suppose that X is a discrete random variable with PMF p_X, where for $x \in R$,

$$p_X(x) = \begin{cases} C_x^3 (0.6)^x (0.4)^{3-x} & \text{if } x \in \{0, 1, 2, 3\}; \\ 0 & \text{otherwise.} \end{cases}$$

   Find $E(X^2)$.

22. (8 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = \frac{1}{2x^2} \cdot I_{(-\infty,-1)\cup(1,\infty)}(x)$$

   for $x \in R$.

   (a) (4 ps) Find a PDF of $1/X$.

   (b) (4 ps) Find $E(1/X)$.

23. (8 pts) Suppose that X and Y are discrete random variables, and

$$P((X,Y) = (x,y)) = \begin{cases} 0.5 & \text{if } (x,y) = (1,2); \\ 0.1 & \text{if } (x,y) = (3,2); \\ 0.3 & \text{if } (x,y) = (3,6); \\ 0.1 & \text{if } (x,y) = (3,7); \\ 0 & \text{otherwise.} \end{cases}$$

   It can be shown that for discrete random variables X and Y,

$$E(g(X,Y)) = \sum_{(x,y):P((X,Y)=(x,y))>0} g(x,y)P((X,Y) = (x,y)) \quad (1)$$

   if g is nonnegative.

   (a) (4 pts) Find $E(XY)$ using (1).

   (b) (4 pts) Find $E(XY)$ using the PMF of XY.

24. (4 pts) Suppose that X has PDF f_X, and g is a function defined by

$$g(x) = \sum_{i=1}^{m} a_i I_{A_i}(x),$$

   where $a_1, \ldots, a_m$ are constants and $A_1, \ldots, A_m$ are disjoint intervals. Therefore, $g(X)$ has only m possible values $a_1, \ldots, a_m$. Verify that

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

25. (8 pts) Suppose that X is a discrete random variable with PMF p_X, where

$$p_X(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!} I_{\{0,1,2,\ldots\}}(x) \tag{2}$$

for $x \in R$, where $\lambda > 0$ is a constant.

(a) (4 pts) Show that $E(X) = \lambda$ and $E(X(X-1)) = \lambda^2$.

(b) (4 pts) Find $Var(X)$. You may use the result in Part (a) even if you choose not to do Part (a).

Note. The distribution of X with the PMF p_X given in (2) is called the Poisson distribution with mean λ.

26. (4 pts) Suppose that X is a random variable with MGF M_X, where

$$M_X(t) = 0.6 + 0.4e^{2t}$$

for $t \in (-\infty, \infty)$. Find $E(X^k)$ for $k \in \{1, 2, 3, 4\}$.

Hint. You may use the following result.

Fact 2 Suppose that

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

for $|x| < r$ for some positive constant r, then

$$f^{(k)}(0) = a_k \cdot k!$$

for $k \in \{0, 1, 2, \ldots\}$.

27. (4 pts) Suppose that Y is a discrete random variable with PMF p_Y, where

$$p_Y(y) = \begin{cases} 0.6 & \text{if } y = 0; \\ 0.4 & \text{if } y = 2; \\ 0 & \text{otherwise.} \end{cases}$$

Show that the distribution of Y is the same as the distribution of the X in Problem 26 by verifying that X and Y have the same MGF.

28. (8 pts) Suppose that X is a random variable whose distribution is the Poisson distribution with mean λ, where $\lambda > 0$. The PMF of X is the function p_X given in (2) in Problem 25.

(a) (4 pts) Find the MGF of X.

(b) (4 pts) Show that $E(X) = \lambda$ and find $Var(X)$ using the MGF of X.

29. (4 pts) Suppose that X is a random variable with MGF M_X, where $M_X(t) < \infty$ for $t \in (-h, h)$ for some positive constant h. For constants a and $b \in R$, let $Y = a + bX$ and let M_Y be the MGF of Y. Show that if $b \neq 0$, then

$$M_Y(t) = e^{ta} M_X(tb) < \infty$$

for $t \in (-h/|b|, h/|b|)$.

30. (12 pts) For constants $\mu \in R$ and $\sigma > 0$, define

$$f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$$

for $x \in R$. Suppose that X is a random variable with PDF $f_{\mu,\sigma}$.

(a) (4 pts) Let $Y = (X - \mu)/\sigma$. Find the MGF of Y. You may use the result that

$$\int_{-\infty}^{\infty} e^{tx} f_{0,1}(x)dx = e^{0.5t^2}$$

for $t \in R$, which has been proved in class.

(b) (4 pts) Find the MGF of X.

(c) (4 pts) Find $E(Y^6)$.

Note. For Part (c), it is easier to find $E(Y^6)$ using Fact 2 than performing direct differentiation of the MGF 6 times.

31. (8 pts) Suppose that (X, Y) is a random vector with CDF $F_{X,Y}$, where for $(x, y) \in R^2$,

$$F_{X,Y}(x, y) = 0.5G(x)G(y) + 0.5(1 - e^{-x})(1 - e^{-y})I_{[0,\infty)}(x)I_{[0,\infty)}(y),$$

and the function G is defined by

$$G(x) = xI_{(0,1)}(x) + I_{[1,\infty)}(x)$$

for $x \in R$.

(a) (4 pts) Find $P(0 < X \leq 1 \text{ and } 1 < Y \leq 2)$.

(b) (4 pts) Find the CDF of X.

32. (4 pts) Suppose that (X, Y, Z) is a random vector with joint CDF F. Show that

$$\begin{aligned}
&P((X, Y, Z) \in (a, b] \times (c, d] \times (e, f]) \\
&= F(b, d, f) - F(b, c, f) - F(a, d, f) + F(a, c, f) \\
&-F(b, d, e) + F(b, c, e) + F(a, d, e) - F(a, c, e),
\end{aligned}$$

where a, b, c, d, e, f are constants such that $a < b$, $c < d$ and $e < f$.

33. (12 pts) Suppose that (X, Y) has PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = cxI_{(0,1)}(x)I_{(0,1)}(y)$$

for $(x, y) \in R^2$ and $c > 0$ is a constant.

(a) (4 pts) Show that $c = 2$.

(b) (4 pts) Find $P(X + 2Y \leq 1)$.

(c) (4 pts) Find a PDF of Y.

34. (4 pts) Suppose that (X, Y) has joint PDF $f_{X,Y}$ and there exist two non-negative functions g and h such that

$$f_{X,Y}(x, y) = g(x)h(y)$$

for $(x, y) \in R^2$. Show that there exist f_X: a PDF of X and f_Y: a PDF of Y such that

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) \tag{3}$$

for $(x, y) \in R^2$.

Note that (3) implies that

$$\begin{aligned}
&P(\{X \in A\} \cap \{Y \in B\}) \\
&= \int_{A\times B} f_X(x) f_Y(y) d(x,y) \\
&= \int_A f_X(x)dx \cdot \int_B f_Y(y)dy \\
&= P(\{X \in A\})P(\{Y \in B\})
\end{aligned}$$

for A, $B \in \mathcal{B}(R)$, which implies that X and Y are independent.

35. (10 pts) Suppose that (X, Y) has PDF $f_{X,Y}$, where

$$f_{X,Y}(x,y) = ce^{-(x^2+y^2)/2} I_{(0,\infty)}(x) I_{(0,\infty)}(y)$$

for $(x, y) \in R^2$ and $c > 0$ is a constant. Let $U = \sqrt{X^2 + Y^2}$ and $V = \tan^{-1}(Y/X)$. Note that for $z \in (-\infty, \infty)$, $\tan^{-1}(z)$ is the value $\theta \in (-\pi/2, \pi/2)$ such that $\tan(\theta) = z$.

(a) (4 pts) Find a PDF of (U, V). Leave the constant c in your answer.

(b) (4 pts) Find a PDF of V. Leave the constant c in your answer.

(c) (2 pts) Find c using your answer in Part (b) and the fact that the integral of a PDF of V over $(-\infty, \infty)$ is 1.

Remark. The result from Problem 35(c) can be used for finding $\int_{-\infty}^{\infty} e^{-x^2/2}dx$. To see this, let $I = \int_0^{\infty} e^{-x^2/2}dx$, then

$$1 = \int_{R^2} f_{X,Y}(x,y)d(x,y) = cI^2,$$

so $I = 1/\sqrt{c}$ and $\int_{-\infty}^{\infty} e^{-x^2/2}dx = 2I = 2/\sqrt{c}$ (you should be able to obtain $2/\sqrt{c} = \sqrt{2\pi}$ if your answer for c is correct).

36. (10 pts) Let Γ be the function on $(0, \infty)$ defined by

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx$$

for $a > 0$. Suppose that α and β are two positive constants and (X, Y) has joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} e^{-x} I_{(0,\infty)}(x) y^{\beta-1} e^{-y} I_{(0,\infty)}(y)$$

for $(x, y) \in R^2$.

(a) (4 pts) Let M be the function on $(-\infty, 1) \times (-\infty, 1)$ defined by

$$M(t_1, t_2) = \left(\frac{1}{1-t_1}\right)^{\alpha} \left(\frac{1}{1-t_2}\right)^{\beta}$$

for $(t_1, t_2) \in (-\infty, 1) \times (-\infty, 1)$. Show that M is the joint MGF of (X, Y).

(b) (2 pts) Find the MGF of X.

(c) (4 pts) Find $E(XY)$ and $E(X)$.

37. (2 pts) Consider the (X, Y) in Problem 36. The distribution of X is called the gamma distribution with shape parameter α and scale parameter 1, denoted by $\Gamma(\alpha, 1)$. Show that the distribution of $(X+Y)$ is $\Gamma(\alpha+\beta, 1)$. Hint: the MGF of $(X+Y)$ can be easily obtained from the MGF of (X, Y).

38. (8 pts) Suppose that (X, Y) has a joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = cI_S(x, y)$$

for $(x, y) \in R^2$,

$$S = \{(x, y) : -2 < x + 2y < 2 \text{ and } -2 < x - 2y < 2\},$$

and $c = 1/\int_{R^2} I_S(x, y)d(x, y)$.

(a) (4 pts) Find $P((X + 2Y) > 0)$. You may leave c in your answer.

(b) (4 pts) Find c.

39. (8 pts) Suppose that (X, Y) has a joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = cI_S(x, y)$$

for $(x, y) \in R^2$,

$$S = \{(x, y) : -2 < x + 2y < 2 \text{ and } -2 < x - 2y < 2\},$$

and $c = 1/\int_{R^2} I_S(x, y)d(x, y)$.

(a) (4 pts) Find $E(X|Y)$. You may leave c in your answer.

(b) (4 pts) Find $Var(X|Y)$. You may leave c in your answer.

40. (16 pts) Suppose that (X, Y) is a discrete random vector with PMF $p_{X,Y}$, where

$$P((X, Y) = (x, y)) = \begin{cases} 0.5 & \text{if } (x, y) = (1, 2); \\ 0.4 & \text{if } (x, y) = (0, -3); \\ 0.1 & \text{if } (x, y) = (1, -3); \\ 0 & \text{otherwise.} \end{cases}$$

(a) (8 pts) Find $E(X|Y = y)$ and $Var(X|Y = y)$ for $y \in \{2, -3\}$.

(b) (8 pts) Find $Var(E(X|Y))$, $E(Var(X|Y))$ and $Var(X)$. Verify the equality

$$Var(X) = Var(E(X|Y)) + E(Var(X|Y))$$

based on your answers.

41. (8 pts) Suppose that (X, Y) has joint PDF $f_{X,Y}$.

(a) (4 pts) For random variables $g(X, Y)$ and $h(Y)$, show that

$$E(g(X, Y)h(Y)|Y) = h(Y)E(g(X, Y)|Y).$$

(b) (4 pts) For random variables $g_1(X, Y)$ and $g_2(X, Y)$, show that

$$E((g_1(X, Y) + g_2(X, Y))|Y) = E(g_1(X, Y)|Y) + E(g_2(X, Y)|Y).$$

42. (16 pts) For $\mu \in R$ and $\sigma > 0$, let $f_{\mu,\sigma}$ be the PDF of $N(\mu, \sigma^2)$ given in Problem 30. Suppose that X and Y are random variables, Y has PDF $f_{0,1}$, and $\{f_{1+2y,1} : y \in R\}$ is a version of the conditional PDF of X given Y.

    (a) (6 pts) Find a PDF of X.

    (b) (6 pts) Find $E(Y|X)$.

    (c) (4 pts) Determine whether X and Y are independent. Justify your answer.

43. (6 pts) Suppose that (X, Y) has a joint PDF and X and Y are independent. Suppose that u and v are functions such that $E(u(X))$ and $E(v(Y))$ are finite. Show that

$$E(u(X)v(Y)) = E(u(X))E(v(Y)).$$

44. (6 pts) Consider the (X, Y) in Problem 36. Determine whehter $X + Y$ and $X - Y$ are indepedent based on the MGF of (X, Y).

45. (4 pts) Consider the (X, Y) in Problem 40. Determine whehter X and Y are independent.

46. (4 pts) Suppose that X and Y are discrete random variables, X has m possible values $x_1, \ldots, x_m$, and Y has n possible values $y_1, \ldots, y_n$. Suppose that for $x \in \{x_1, \ldots, x_m\}$,

$$P(X = x|Y = y_1) = P(X = x|Y = y_2) = \cdots = P(X = x|Y = y_n).$$

Show that X and Y are independent.

47. (8 pts) Suppose that Z_1, Z_2, Z_3 are independent random variables and $Z_i \sim N(0, 1)$ for $i = 1, 2, 3$. Let

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}.$$

    (a) (4 pts) Find a PDF of (Y_1, Y_2, Y_3).

    (b) (4 pts) Determine whether Y_1 and (Y_2, Y_3) are independent. Justify your answer.

48. (6 pts) Suppose that $X_1, \ldots, X_n$ are IID and $X_1 \sim N(\mu, \sigma^2)$. Let $\bar{X} = \sum_{i=1}^n X_i/n$ and $Y = (X_1 - \bar{X}, \ldots, X_n - \bar{X})^T$. Show that $\bar{X}$ and Y are independent.

49. (12 pts) Suppose that $Z_1, \ldots, Z_m$ are IID and $Z_1 \sim N(0, 1)$. Let $U = \sum_{i=1}^m Z_i^2$.

    (a) (6 pts) Show that $U/2 \sim \Gamma(m/2, 1)$.

    Note.

    - Recall that for $\alpha > 0$, the MGF of the gamma distribution $\Gamma(\alpha, 1)$ is the MGF of the random variable X in Problem 36.
    - To solve this problem, you may use the results in Problem 36.
    - For $\mu \in R$ and $\sigma > 0$, a PDF of $N(\mu, \sigma^2)$ is the $f_{\mu,\sigma}$ given in Problem 30.

(b) (6 pts) Let

$$f_U(x) = \frac{1}{2^{m/2}\Gamma(m/2)} x^{m/2-1} e^{-x/2} I_{(0,\infty)}(x)$$

for $x \in R$, where Γ is defined in Problem 36. Show that f_U is a PDF of U.

Hint: find a PDF of $U/2$ by finding a PDF of the random variable X in Problem 36, and then derive a PDF of U using the PDF of $U/2$.

Note. In Problem 49, the distirbution of U is called the chi-squared distribution with m degrees of freedom, denoted by $\chi^2(m)$.

50. (6 pts) Suppose that Z and W are independent random variables, $Z \sim N(0, 1)$ and $W \sim \chi^2(m)$. Let

$$T = \frac{Z}{\sqrt{W/m}},$$

then the distribution of T is called the t distribution with m degrees of freedom, denoted by $t(m)$. Find a PDF of T.

51. (6 pts) Suppose that (X, Y) is a vector of two discrete random variables with joint PMF $p_{X,Y}$, where

$$p_{X,Y}(x, y) = \begin{cases} 0.5 & \text{if } (x, y) = (1, 2); \\ 0.4 & \text{if } (x, y) = (0, a); \\ 0.1 & \text{if } (x, y) = (-1. - 2); \\ 0 & \text{otherwise,} \end{cases}$$

and a is a constant.

(a) (4 pts) Express $Corr(X, Y)$ as a a function of a.

(b) (2 pts) Find all a's such that $|Corr(X, Y)| = 1$.

52. (6 pts) Suppose that X and Y are random variables such that $Var(X)$ and $Var(Y)$ are both finite and $Var(X) > 0$. Define a function S by

$$S(a, b) = E(Y - (a + bX))^2$$

for $(a, b) \in R^2$. Show that S is minimized when $(a, b) = (a_0, b_0)$, where $b_0 = Cov(X, Y)/Var(X)$ and $a_0 = E(Y) - b_0E(X)$. Also, show that

$$S(a_0, b_0) = \frac{Var(X)Var(Y) - (Cov(X, Y))^2}{Var(X)}. \tag{4}$$

53. (4 pts) Suppose that X and Y are random variables, f_Y is a PDF of Y, $S_Y = \{y : f_Y(y) > 0\}$, and $\{f_{X|Y=y} : y \in S_Y\}$ is a version of the conditional PDF of X given Y. Then for g such that $E(g(X, Y))$ is finite, $E(g(X, Y)|Y)$ can be obtained using

$$E(g(X, Y)|Y = y) = \int g(x, y) f_{X|Y=y}(x) dx \tag{5}$$

for all $y \in S_Y$. Use (5) to show that $E(XY|Y) = YE(X|Y)$ when $E(XY)$ and $E(X)$ are finite.

54. (22 pts) Suppose that (X, Y, Z) is a random vector with joint PDF $f_{X,Y,Z}$, where

$$f_{X,Y,Z}(x, y, z) = ce^{-(x^2+4xy+5y^2)} ze^{-z} I_{(0,\infty)}(z)$$

for $(x, y, z) \in R^3$, and $c > 0$ is a constant.

(a) (2 pts) Find a version of the conditional PDF of Z given (X, Y).

(b) (6 pts) Find c, a PDF of Y, and a version of the conditional PDF of X given Y.

(c) (6 pts) Find $E(X|Y)$, $E(X^2|Y)$ and $Var(X|Y)$.

(d) (2 pts) Find the best linear predictor of X based on Y.

(e) (8 pts) Find $Cov(X, Y)$ and $Corr(X, Y)$.

55. (4 pts) Suppose that X and Y are random variables such that $Y \sim N(0, 1)$ and a version of the conditional PDF of X given Y is $\{g_y : y \in R\}$, where g_y is the function $f_{\mu,\sigma}$ given in Problem 30 with $\mu = y$ and $\sigma = 1$. Find a version of the conditional PDF of Y given X.

56. (4 pts) Suppose that W is an $n \times m$ matrix of random variables, and B is an $m \times k$ non-random matrix. Show that $E(WB) = E(W)B$.

57. (4 pts) Suppose that $\boldsymbol{X} = (X_1, \ldots, X_m)^T$ is a random vector. Let Σ be the covariance matrix of $\boldsymbol{X}$. Suppose that A is a $n \times m$ non-random matrix. Show that the covariance matrix of $A\boldsymbol{X}$ is $A\Sigma A^T$. You may use the fact that for a random vector $\boldsymbol{Z}$, the covariance matrix of $\boldsymbol{Z}$ is $E(\boldsymbol{Z}_*\boldsymbol{Z}_*^T)$, where $\boldsymbol{Z}_* = \boldsymbol{Z} - E(\boldsymbol{Z})$.

58. (6 pts) Suppose that (X, Y) is a random vector with covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 1 & a \\ a & 0.25 \end{pmatrix}$$

and a is a constant.

(a) (2 pts) Express $Corr(X, Y)$ as a function of a.

(b) (4 pts) Find a constant b such that $Var(X - bY) = 0$ when $a = 0.5$.

59. (4 pts) Suppose that $\boldsymbol{X} = (X_1, X_2, X_3, X_4)^T$ is a random vector with covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Find $Var(X_1 + X_2 + X_3)$.

Hint: Apply Fact 2 in the handout "Covariance and correlation".

60. (3 pts) Suppose that U and X are random variables such that $E(X^2) < \infty$ and $E(U^2) < \infty$. Let $S(b) = E(U - bX)^2$, where b is a constant in R. Verify that

$$\frac{d}{db} S(b) = E\left[\frac{d}{db}(U - bX)^2\right].$$

Remark. The result in Problem 60 implies that

$$\frac{\partial}{\partial b_i} E[(Y-(a+b_1X_1+\cdots+b_kX_k))^2] = E\left[\frac{\partial}{\partial b_i}(Y-(a+b_1X_1+\cdots+b_kX_k))^2\right]$$

for $i \in \{1, \ldots, k\}$ and

$$\frac{\partial}{\partial a} E[(Y-(a+b_1X_1+\cdots+b_kX_k))^2] = E\left[\frac{\partial}{\partial a}(Y-(a+b_1X_1+\cdots+b_kX_k))^2\right].$$

61. (6 pts) Suppose that $Z_1, \ldots, Z_n$ are independent random variables and $Z_i \sim N(0,1)$ for $i = 1, 2, \ldots, n$. Suppose that A is an $n \times n$ invertible matrix of constants in R and $\mu_1, \ldots, \mu_n$ are constants in R. Let

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = A \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}.$$

Find a PDF of $(Y_1, \ldots, Y_n)$ that is determined by AA^T and $\mu_1, \ldots, \mu_n$. You may use results from linear algebra such as $\det(B^T) = \det(B)$ and

$$\det(BC) = \det(B) \cdot \det(C)$$

for square matrices B and C. Here $\det(A)$ denotes the determinant of a square matrix A.

Note. In Problem 61, AA^T and $(\mu_1, \ldots, \mu_n)^T$ are the covariance matrix and the mean vector of $(Y_1, \ldots, Y_n)^T$, respectively.

62. (6 pts) Prove the following result:

Fact 3 Suppose that the distribution of $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)^T$ is multivariate normal. If $Cov(X_i, Y_j) = 0$ for $1 \le i \le m$, $1 \le j \le n$, then the two random vectors $(X_1, \ldots, X_m)^T$ and $(Y_1, \ldots, Y_n)^T$ are independent.

63. (6 pts) Suppose that

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 & 2 & 5 \\ 2 & 16 & 3 \\ 5 & 3 & 25 \end{pmatrix}\right).$$

(a) (3 pts) Find a constant b such that $Y - bX$ is independent of X.

(b) (3 pts) Find constants c and d such that $Z - cY - dX$ is independent of (X, Y).

64. (15 pts) Suppose that $\boldsymbol{X} = (X_1, X_2, X_3, X_4)^T$ is a random vector with $E(\boldsymbol{X}) = (0, 0, 0, 0)^T$ and covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Suppose that $\boldsymbol{X} \sim N(E(\boldsymbol{X}), \Sigma)$.

(a) (3 pts) Find the best linear predictor of $(X_1, X_4)^T$ based on X_2 and X_3.

(b) (3 pts) Find a version of the conditional PDF of X_1 given X_2.

(c) (3 pts) Find a version of the conditional PDF of X_1 given $(X_2, X_3, X_4)^T$.

(d) (3 pts) Find a version of the conditional PDF of $(X_1, X_2)^T$ given $(X_3, X_4)^T$.

(e) (3 pts) Find $E(X_1X_2|X_3, X_4)$.

65. (6 pts) Suppose that $\boldsymbol{Y}$, $\boldsymbol{X}$ and $\boldsymbol{U}$ are random vectors in R^m, R^n, R^m respectively such that $\boldsymbol{X}$ and $\boldsymbol{U}$ are independent and

$$\boldsymbol{Y} = g(\boldsymbol{X}) + \boldsymbol{U} \tag{6}$$

for some function $g : R^n \to R^m$. Suppse that g is differentiable, $\boldsymbol{X}$ has a PDF $f_{\boldsymbol{X}}$ that is positive on R^n, and $\boldsymbol{U}$ has a PDF $f_{\boldsymbol{U}}$ that is positive on R^m. For $\boldsymbol{x} \in R^n$, define

$$f_{\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}}(\boldsymbol{y}) = f_{\boldsymbol{U}}(\boldsymbol{y} - g(\boldsymbol{x})) \tag{7}$$

for $\boldsymbol{y} \in R^m$. Show that $\{f_{\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}} : \boldsymbol{x} \in R^n\}$ is a version of the conditional PDF of $\boldsymbol{Y}$ given $\boldsymbol{X}$.

66. (4 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$, where $\mu \in R$ and $\sigma > 0$ are unknown. Let $\bar{X} = \sum_{i=1}^n X_i/n$, $\bar{Y} = \sum_{i=1}^n X_i^2/n$, and $\bar{Z} = \sum_{i=1}^n (X_i - \mu)^2/n$. Which of the following statements are true? You may write down your answers directly without justification.

(a) $\bar{X}$ is a consistent estimator of μ.

(b) $\bar{Y} - (\bar{X})^2$ is a consistent estimator of σ^2.

(c) $\bar{Z}$ is a consistent estimator of σ^2.

(d) $\bar{Z}$ converges to σ^2 in probability as $n \to \infty$.

67. (4 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $U(0, \theta)$, where $\theta > 0$. Find a consistent estimator of $1/\theta$ and justify your answer.

68. (8 pts) Suppose that $((X_1, Y_1), \ldots, (X_n, Y_n))$ is a random sample from the distribution of a random vector (X, Y), where

$$Y = a + bX + \varepsilon,$$

a, b are constants, X and ε are independent, $E(\varepsilon) = 0$ and $Var(\varepsilon) = \sigma^2 < \infty$. Let $\bar{X}$ and $\bar{Y}$ be the sample means of $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ respectively, let

$$S_X^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2,$$

$$\hat{b} = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{(n-1)S_X^2},$$

and $\hat{a} = \bar{Y} - \hat{b}\bar{X}$. Show that $(\hat{a}, \hat{b})$ is a consistent estimator of (a, b) assuming that $E(X^2) < \infty$, which implies that $E(X)$ is finite.

69. (6 pts) Prove the following result.

Fact 4 Suppose that $\{X_{1,n}\}_{n=1}^{\infty}$, …, $\{X_{k,n}\}_{n=1}^{\infty}$ are k sequences of random variables on the same probability space, and $X_{1,n} \xrightarrow{\mathcal{P}} Y_1$, …, $X_{k,n} \xrightarrow{\mathcal{P}} Y_k$ for some random vector $(Y_1, \ldots, Y_k)^T$. Then

$$(X_{1,n}, \ldots, X_{k,n})^T \xrightarrow{\mathcal{P}} (Y_1, \ldots, Y_k)^T.$$

Hint: make use of the equality:

$$\|(X_{1,n}, \ldots, X_{k,n})^T - (Y_1, \ldots, Y_k)^T\|^2 = |X_{1,n} - Y_1|^2 + \cdots + |X_{k,n} - Y_k|^2.$$

70. (6 pts) Suppose that $\theta \in R$ is a paramater to be estimated, and $\hat{\theta}$ is an estimator of θ such that

$$E(\hat{\theta}) = \theta$$

and

$$\lim_{n\to\infty} Var(\hat{\theta}) = 0.$$

Show that $\hat{\theta}$ is a consistent estimator of θ.

Hint: make use of Chebyshev's inequality:

$$P(|X - E(X)| > k) \leq \frac{Var(X)}{k^2}$$

for a positive constant $k > 0$, or Markov's inequality: for a nonnegative random variable X,

$$P(X > M) \leq \frac{E(X)}{M}$$

for a positive constant M.

71. (6 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from a discrete distribution with three possible values a_1, a_2, a_3. Let

$$p_j = P(X_1 = a_j)$$

and

$$\hat{p}_j = \frac{1}{n}\sum_{i=1}^{n} I_{\{a_j\}}(X_i)$$

for $j = 1, 2, 3$. Let

$$Y_n = \sqrt{n}\left(\begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}\right).$$

Find the limiting distribution of Y_n. That is, find the distribution D_0 such that Y_n converges to D_0 in distribution as $n \to \infty$.

72. (6 pts) Consider the following result, which has been proved in class:

Fact 5 Suppose that $\boldsymbol{U} \sim N(\boldsymbol{0}, \Sigma)$ and $\Sigma^2 = \Sigma$. Then $\boldsymbol{U}^T\boldsymbol{U} \sim \chi^2(k)$, where $k = \text{trace}(\Sigma)$.

Suppose that $Z_1, \ldots, Z_n$ are IID $N(0,1)$ random variables. Let $\bar{Z} = \sum_{i=1}^n Z_i/n$. Apply Fact 5 to show that $\sum_{i=1}^n (Z_i - \bar{Z})^2 \sim \chi^2(n-1)$.

Hint: note that the distribution of $(Z_1 - \bar{Z}, \ldots, Z_n - \bar{Z})$ is a multivariate normal distribution.

73. (12 pts) Consider the $\hat{p}_1$ in Problem 71. Suppose that $p_1 \in (0,1)$.
    (a) (3 pts) Find the limiting distribution of $\sqrt{n}(\hat{p}_1 - p_1)/\sqrt{p_1(1-p_1)}$. Justify your answer.
    (b) (3 pts) Find the limiting distribution of $n(\hat{p}_1 - p_1)^2/(p_1(1-p_1))$. Justify your answer.
    (c) (3 pts) Find the limiting distribution of $n(\hat{p}_1 - p_1)^2/(\hat{p}_1(1-\hat{p}_1))$. Justify your answer.
    (d) (3 pts) Suppose that $\{a_n\}_{n=1}^\infty$ is a sequence of postive constants such that $\lim_{n\to\infty} a_n/\sqrt{n} = 0$. Show that $a_n(\hat{p}_1 - p_1)$ converges to 0 in distribution as $n \to \infty$.

74. (6 pts) Suppose that $\{X_n\}_{n=1}^\infty$ is a sequence of random variables and $X_n \overset{\mathcal{D}}{\to} c$ as $n \to \infty$, where c is a constant.
    (a) (2 pts) Let F_n be the CDF of X_n. What can be said about $\lim_{n\to\infty} F_n(x)$ for $x \neq c$?
    (b) (4 pts) Show that $X_n \overset{\mathcal{P}}{\to} c$ as $n \to \infty$ based on the result from Part (a).

75. (4 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$. Let $\bar{Y} = \sum_{i=1}^n X_i^2/n$. Find the limiting distribution of $\sqrt{n}(\bar{Y} - E(X_1^2))$. Justify your answer.