Homework Problems

Note.

- Always show your work in your homework solutions to receive full points unless it is stated otherwise.

1. (5 pts) Suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of random variables and c is a constant. Show that if X_n converges to c in distribution as $n \to \infty$, then X_n converges to c in probability as $n \to \infty$.

2. (20 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $Exp(\theta)$ (the exponential distribution with expectation θ), where $\theta > 0$. Let $\bar{X} = \sum_{i=1}^{n} X_i/n$ be the sample mean.

   Let $\hat{V}$ be a consistent estimator of $Var(X_1)$.

   (a) (5 pts) Find a consistent estimator of $Var(X_1)$.

   (b) (5 pts) Let $\hat{V}_1$ be the consistent estimator of $Var(X_1)$ found in Part (a). Find the limiting distribution of $\sqrt{n}(\bar{X} - \theta)/(\hat{V}_1/\sqrt{V})$.

   (c) (5 pts) Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real nubmers such that $\lim_{n\to\infty} a_n = 1$. Find the limiting distribution of $a_n\sqrt{n}(\bar{X} - \theta)/\sqrt{\hat{V}}$.

   (d) (5 pts) Find the limiting distribution of

   $$\frac{\sqrt{n}(\bar{X} - \theta)}{\sqrt{\hat{V}}} + (\hat{V} - \theta)^2.$$

   Note. You may use the fact that $Var(X) = 1$ if $X \sim Exp(1)$.

   Notation. From now on, we will use the notation $\stackrel{\mathcal{D}}{\to}$ to denote convergence in distribution. That is, $X_n \stackrel{\mathcal{D}}{\to} X$ means X_n converges to X in distribution.

3. (4 pts) Determine whether the following statements are true. You may write down your answers directly without justification.

   (a) Suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of random variables and X is a random variable. If X_n converges to X in probability as $n \to \infty$, then $X_n \stackrel{\mathcal{D}}{\to} X$ as $n \to \infty$.

   (b) Suppose that $\hat{\theta}$ is a consistent estimator of θ, then $\hat{\theta}^2$ is a consistent estimator of θ^2.

   (c) Suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of random variables and X is a random variable such that

   $$\sqrt{n}(X_n - 2) \stackrel{\mathcal{D}}{\to} X$$

   as $n \to \infty$. Then

   $$\sqrt{n}(X_n^3 - 2^3) \stackrel{\mathcal{D}}{\to} 12X$$

   as $n \to \infty$.

   (d) The statement in Part (c) still holds if all the $\sqrt{n}$'s are replaced by $n^{1/3}$'s.

4. (5 pts) Suppose that X is a random variable such that $X \sim N(0,1)$. Let $X_n = (-1)^n X$ for $n \in \{1, 2, \ldots\}$. Show that $X_n \xrightarrow{\mathcal{D}} X$ as $n \to \infty$ and X_n does not converge to X in probability as $n \to \infty$.

5. (15 pts) For $\alpha > 0$, $\beta > 0$, the gamma distribution $\Gamma(\alpha, \beta)$ is the distribution with PDF $f_{\alpha,\beta}$, where

$$f_{\alpha,\beta}(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} I_{(0,\infty)}(x)$$

for $x \in (-\infty, \infty)$ and

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Suppose that $(X_1, \ldots, X_n)$ is a random sample from $\Gamma(\alpha, \beta)$, where $\alpha > 0$, $\beta > 0$. Let $\bar{X} = \sum_{i=1}^n X_i/n$ be the sample mean.

   (a) (3 pts) Show that $\Gamma(\theta + 1) = \theta\Gamma(\theta)$ for every $\theta > 0$.

   (b) (6 pts) Show that $E(X_1) = \alpha\beta$ and $Var(X_1) = \alpha\beta^2$.

   (c) (3 pts) Find the limiting distribution of $\sqrt{n}(\bar{X} - \alpha\beta)$.

   (d) (3 pts) Find the limiting distribution of $\sqrt{n}(e^{\bar{X}} - e^{\alpha\beta})$.

6. (9 pts) In Problem 5, suppose that we are given additional information that $\alpha = 2$.

   (a) (4 pts) Find the limiting distribution of

$$\frac{\sqrt{n}(\bar{X} - 2\beta)}{|\bar{X}|}.$$

   (b) (5 pts) For $\alpha \in (0, 1)$, find an approximate $(1-\alpha)$ confidence interval of β based on the result of Part (a).

7. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$, where $\sigma > 0$. Let $\bar{X} = \sum_{i=1}^n X_i/n$ be the sample mean and let $S = \sqrt{\sum_{i=1}^n (X_i - \bar{X})^2/(n-1)}$ be the sample standard deviation. Use the fact that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$

to construct a 95% confidence interval of σ. Note.

   - For $\alpha \in (0, 1)$ and a positive integer m, we use $k_{\alpha,m}$ to denote the $(1-\alpha)$ quantile of the $\chi^2(m)$ distribution, that is,

$$P(\chi^2(m) > k_{\alpha,m}) = \alpha.$$

     You may use the quantiles of $\chi^2(n-1)$ in the expression of your confidence interval.

   - The center of your confidence interval does not have to be S.

8. (20 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from a distribution with MGF (moment generating function) M_X and

$$M_X(t) = 0.25 + 0.5e^t + 0.25e^{2t}$$

for $t \in (-\infty, \infty)$. Let $\bar{X} = \sum_{i=1}^n X_i/n$, $\bar{Y} = \sum_{i=1}^n X_i^2/n$ and

$$S = \sqrt{\frac{n\bar{Y} - n(\bar{X})^2}{n-1}}.$$

(a) (5 pts) Show that $E(X_1^k) = 0.5 + 0.25 \cdot 2^k$ for every positive integer k.

(b) (10 pts) Let $\sigma = \sqrt{Var(X_1)}$ and let $\hat{\sigma} = S \cdot \sqrt{(n-1)/n}$. Find σ and the limiting distribution of $\sqrt{n}(\hat{\sigma} - \sigma)$. You may use the result from Part (a) even if you choose not to do Part (a).

(c) (5 pts) Find the limiting distribution of $\sqrt{n}(S - \sigma)$.

9. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $\Gamma(\alpha, \beta)$, where $\alpha > 0$, $\beta > 0$. Find a MME (method of moment estimator) of (α, β). You may use the result from Problem 5.

10. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$, where $\sigma = 2$ is known and $\mu \in (-\infty, \infty)$ is an unknown parameter. Let $\bar{X} = \sum_{i=1}^n X_i/n$. Show that the MLE (maximum likelihood estimator) of μ is $\bar{X}$.

11. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $Bin(1, \theta)$, where $\theta \in \{0.1, 0.2, 0.3\}$. Suppose that $n = 100$ and we observe $(X_1, \ldots, X_{40}) = (0, \ldots, 0)$ and $(X_{41}, \ldots, X_{100}) = (1, \ldots, 1)$. Find the observed MLE of θ. You may use the following results.

- The R output after running the command

  ```
  theta <- c(0.1, 0.2, 0.3)
  ( theta^(60) )*( (1-theta)^(40) )
  ```

  is

  ```
  [1] 1.478088e-62 1.532496e-46 2.698963e-38
  ```

- The R output after running the command

  ```
  theta <- c(0.1, 0.2, 0.3)
  ( theta^(40) )*( (1-theta)^(60) )
  ```

  is

  ```
  [1] 1.797010e-43 1.684997e-34 6.176360e-31
  ```

12. (5 pts) Suppose that Θ is a set and for each $\eta \in \Theta$, f_η is a PMF such that

$$\sum_{x=1}^{\infty} f_\eta(x) = 1.$$

Suppose that X_1 is a discrete random variable with PMF f_θ for some $\theta \in \Theta$. Apply Jensen's inequality to show that

$$E[\log f_\eta(X_1)] \leq E[\log f_\theta(X_1)]$$

for every $\eta \in \Theta$. You may use the fact that if φ is a real-valued function defined on an open interval I and $\varphi'' > 0$ on I, then φ is a convex function on I.

13. (20 pts) For $\eta > 0$, let

$$f_\eta(x) = \frac{1}{\eta} \frac{e^{-x/\eta}}{e^{-1}} I_{[\eta,\infty)}(x)$$

for $x \in R$. Suppose that $(X_1, \ldots, X_n)$ is a random sample from a distribution with PDF f_θ, where $\theta > 0$.

(a) (5 pts) Find a MME of θ.

(b) (5 pts) Find the MSE of the MME in Part (a).

(c) (5 pts) Find the MLE of θ.

(d) (5 pts) Show that the MSE of the MLE in Part (c) is less than or equal to the MSE of the MME in Part (a) for large enough n.

14. (15 pts) Suppose that $(X_1, \ldots, X_n)$ is a sample from $N(\mu, \sigma^2)$, where $\mu \in (-\infty, \infty)$ and $\sigma > 0$. Let $\bar{X} = \sum_{i=1}^n X_i/n$ and $S = \sqrt{\sum_{i=1}^n (X_i - \bar{X})^2/(n-1)}$.

(a) (5 pts) Show that $(\bar{X}, S)$ is a sufficient statistic of (μ, σ).

(b) (5 pts) Suppose that we use $(\bar{X})^2$ to estimate μ^2. Find the bias of $(\bar{X})^2$.

(c) (5 pts) Show that $(\bar{X})^2 - (S^2/n)$ is an unbiased estimator of μ^2.

15. (10 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $\Gamma(\alpha, \beta)$, where $\alpha > 0$, $\beta > 0$. Let $\bar{X} = \sum_{i=1}^n X_i/n$ and $\bar{Y} = \sum_{i=1}^n \log(X_i)/n$.

(a) (5 pts) Show that $(\bar{X}, \bar{Y})$ is a sufficient statistic for (α, β).

(b) (5 pts) Suppose that α is known. Show that the MLE of β is $\bar{X}/\alpha$.

16. (5 pts) In Problem 15, suppose that $n = 1000$, $\alpha < 10$ and we observe $(\bar{X}, \bar{Y}) = (1.998, 0.439)$.

(a) (2 pts) Suppose that we ran the following R commands to define two functions in R

```
minus.log.lik <- function(a,b){
  x.bar <- 1.998
  y.bar <- 0.439
  log.lik <- -n*log(gamma(a)) - n*a*log(b) + (a-1)*n*y.bar - n*x.bar/b
  return(-log.lik)
}

f <- function(a){
   x.bar <- 1.998
   return( minus.log.lik(a, x.bar/a) )
}
```

and then we ran the R command `optimize(f, c(0,10))`, which gives the output

```
> optimize(f, c(0, 10))
$minimum
[1] 2.126792

$objective
[1] 1557.931
```

Find the observed MLE for α based on the R output.

(b) (3 pts) Find the observed MLE for β.

Note. You may skip this problem if you are not familiar with R.

17. (15 pts) Suppoe that $(X_1, \ldots, X_n)$ is a random sample from $Bin(1, \theta)$, where $\theta \in \Theta$ and Θ is a subset of $(0, 1)$. Let $\bar{X} = \sum_{i=1}^n X_i/n$ be the sample mean.

    (a) (5 pts) Is $\bar{X}$ a sufficient statistic for θ? Justify your answer.

    (b) (5 pts) Suppose that $\Theta = \{0.3, 0.7\}$. Is $\bar{X}$ a complete statistic for θ? Justify your answer.

    (c) (5 pts) Suppose that $\Theta = \{0.3, 0.5, 0.7\}$. Is $\bar{X}$ a complete statistic for θ? Justify your answer.

18. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $Exp(\lambda)$, the exponential distribution with mean λ, where $\lambda > 0$. Find a complete and sufficient statistic for λ.

19. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $N(\mu, 1)$, where $\mu \in R$ and $n \geq 3$. Find $E(X_3|X_1, X_2)$ and determine whether $E(X_3|X_1, X_2)$ is an unbiased estimator of μ. Justify your answer.

20. (5 pts) In Problem 18, find a MVUE of 2λ.

21. (5 pts) State and prove Basu's theorem.

22. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$, where $\mu \in R$ and $\sigma > 0$ are unknown parameters and $n \geq 3$. Let $\bar{X} = \sum_{i=1}^n X_i/n$ and $S = \sqrt{\sum_{i=1}^n X_i^2/(n-1)}$. For each of the following statistics, determine whether it is a minimum sufficient statistic for (μ, σ). You may write down the final answer only without justification.

    (a) $\sum_{i=1}^n X_i$

    (b) $(X_1, \ldots, X_n)$

    (c) $(\bar{X}, \sum_{i=1}^n X_i^2)$

    (d) $(\bar{X}, S)$

    (e) $(\bar{X}, S, \sum_{i=1}^n X_i^3)$

23. (5 pts) Recall that

$$E[(Y - g(X))^2] = E[(Y - E(Y|X))^2] + E[(E(Y|X) - g(X))^2]. \qquad (1)$$

Apply (1) to show that

$$Var(Y) \geq Var(E(Y|X))$$

and $Var(Y) = Var(E(Y|X))$ implies that $P(Y = E(Y|X)) = 1$. You may use the fact that $E(X^2) = 0$ implies that $P(X = 0) = 1$. Here we assume that all the expectations involved are finite.

24. (10 pts) Suppose that S is a one dimensional complete and sufficient statistic for $\theta \in \Theta$. Let $\varphi(x) = e^x/(1 + e^x)$ for $x \in R$, then $\varphi(S)$ is also a one dimensional complete and sufficient statistic. Suppose that T is a sufficient statistic for $\theta \in \Theta$.

    (a) (5 pts) Show that $P(\varphi(S) = E[E(\varphi(S)|T)|S]) = 1$.

    (b) (5 pts) Apply the result in Problem 23 to deduce that

    $$P(\varphi(S) = E(\varphi(S)|T)) = 1.$$

    Note. The result of this Problem implies that a one dimensional complete and sufficient statistic is minimum sufficient. The result can be generalized to the case where the dimension of S is greater than 1.

25. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from a distribution with PDF f. Let $(X_{(1)}, \ldots, X_{(n)})$ be the order statistic based on $(X_1, \ldots, X_n)$. Let

    $$S_0 = \{(x_1, \ldots, x_n) \in R^n : x_1 < x_2 < \cdots < x_n\}$$

    and define

    $$g(x_1, \ldots, x_n) = n! I_{S_0}(x_1, \ldots, x_n) \prod_{i=1}^{n} f(x_i)$$

    for $(x_1, \ldots, x_n) \in R^n$. We have shown in class that g is a PDF of $(X_{(1)}, \ldots, X_{(n)})$. It can be shown that for (s, t) such that $1 \le s < t \le n$, a PDF of $(X_{(s)}, X_{(t)})$ is the function h defined by

    $$h(u, v) = \frac{n! f(u) f(v) I_S(u, v)}{(s-1)!(t-s-1)!(n-t)!} (F(u))^{s-1} (F(v) - F(u))^{t-s-1} (1 - F(v))^{n-t}$$

    for $(u, v) \in R^2$, where

    $$S = \{(u, v) \in R^2 : u < v\}.$$

    Verify that the function h with $(s, t) = (1, 2)$ is a PDF of $(X_{(1)}, X_{(2)})$ using the joint PDF g.

26. (20 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $U(0, \theta)$, where $\theta > 0$. Let $(X_{(1)}, \ldots, X_{(n)})$ be the order statistic based on $(X_1, \ldots, X_n)$.

    (a) (10 pts) Write down a joint PDF of $(X_{(1)}, X_{(n)})$ using the h in Problem 25, and then find a PDF of $X_{(1)}$ and a PDF of $X_{(n)}$ using the joint PDF of $(X_{(1)}, X_{(n)})$.

    (b) (10 pts) Find $Var(X_{(1)} + X_{(n)})$ and $Var(E[(X_{(1)} + X_{(n)})|X_{(n)}])$. Verify that

    $$Var(E[(X_{(1)} + X_{(n)})|X_{(n)}]) \le Var(X_{(1)} + X_{(n)})$$

    based on your answers.

27. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $U(-\theta, \theta)$, $\theta > 0$. Let $(X_{(1)}, \ldots, X_{(n)})$ be the order statistic based on $(X_1, \ldots, X_n)$. Show that $(X_{(1)}, X_{(n)})$ is a sufficient statistic for θ but not a complete statistic for θ.

28. (10 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $U(-\theta, 0)$, $\theta > 0$. Let $(X_{(1)}, \ldots, X_{(n)})$ be the order statistic based on $(X_1, \ldots, X_n)$. It can be shown that $X_{(1)}$ is a complete statistic for θ.

    (a) (5 pts) Show that $X_{(1)}$ is a sufficient statistic for θ.

    (b) (5 pts) Use the result in Part (a) and the fact that $X_{(1)}$ is a complete statistic for θ to find an MVUE for θ.

29. (5 pts) Prove the Neyman-Pearson lemma for the case where the sample $(X_1, \ldots, X_n)$ is discrete with PMF f.

30. (5 pts) In Problem 18, find a most powerful level α test for the testing problem

$$H_0 : \lambda = 1 \text{ v.s. } H_1 : \lambda = 2.$$

    Here it is assumed that $\lambda \in \{1, 2\}$.

31. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$, where $\sigma > 0$ is known. Consider the testing problem

$$H_0 : \mu \leq 1 \text{ v.s. } H_1 : \mu > 1.$$

    Let

$$Z = \frac{\sqrt{n}(\bar{X} - 1)}{\sigma},$$

    where $\bar{X} = \sum_{i=1}^{n} X_i/n$. The z test rejects H_0 at level α if and only if $Z > z_\alpha$, where z_α is the $(1 - \alpha)$ quantile of $N(0, 1)$. That is, $P(N(0, 1) > z_\alpha) = \alpha$. Show that the size of the z test is α.

32. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$. Construct a level 0.05 test for testing

$$H_0 : \sigma = 1 \text{ v.s. } H_1 : \sigma \neq 1$$

    using the result in Problem 7.

33. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $\Gamma(a, 1)$, where $a > 0$. Let

$$f_a(x) = \frac{1}{\Gamma(a)} x^{a-1} e^{-x} I_{(0,\infty)}(x)$$

    for $x \in R$, then f_a is a PDF of $\Gamma(a, 1)$. Find an approximate most powerful level α test for the testing problem

$$H_0 : a = 2 \text{ v.s. } H_1 : a = 3.$$

    Here it is assumed that $a \in \{2, 3\}$. For $k \in \{1, 2\}$ and a function g, let

$$\mu_0(k, g) = \int_0^\infty f_2(x)(g(x))^k dx$$

    and

$$\mu_1(k, g) = \int_0^\infty f_3(x)(g(x))^k dx.$$

    Let z_α be the $(1 - \alpha)$ quantile of $N(0, 1)$. You may express the rejection region of the test using z_α, $\mu_0(k, g)$ and $\mu_1(k, g)$ with specified k and g.

34. (5 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$. Find a LRT (likelihood ratio test) of approximate size α for testing

$$H_0 : \sigma = 1 \text{ v.s. } H_1 : \sigma \neq 1.$$

Here we assume that the regularity conditions for χ^2 approximation of the distribution of the test statistic of LRT hold and you may express the rejection region of the test using quantiles of χ^2 distributions.

35. (5 pts) For each of the following sets, determine whether the set contains a nonempty open set in R^2. You may write down the final answers only.

    (a) $\{(2t, 3t^2) : t \in (0, 1)\}$

    (b) $(1, 2) \times (3, 4)$

    (c) $\{(0, t) : t \in (0, \infty)\}$

    (d) $\{(x, y) : (x-1)^2 + (y-2)^2 \leq 1\}$

    (e) $\{(x, y) : x + y = 1\}$