Homework Problems

- Note. Always show your work in your homework solutions to receive full points unless it is stated otherwise.

1. (2 pts) Suppose that $\mathcal{F}$ is a σ-field on a sample space Ω and A, B are two events in $\mathcal{F}$. Show that if $A \subset B$, then $P(A) \leq P(B)$.

2. (2 pts) Suppose that $\Omega = \{1, 2, 3, 4, 5\}$ is a sample space. Let $A = \{1, 2, 3\}$, $B = \{1, 2, 4, 5\}$ and $C = \{4, 5\}$. Suppose that $\mathcal{F}$ is a σ-field on Ω such that A, B, C are in $\mathcal{F}$, and P is a real-valued function defined on $\mathcal{F}$. In which of the following cases, P cannot be a probability function? Justify your answer. Note that if P cannot be a probability function for two or more cases, you should list all these cases. For cases where you know that P can be a probability function, you do not need to provide justification.
   (a) $P(A) = 0.2$, $P(B) = 0.3$, $P(C) = 0.2$
   (b) $P(A) = 0.2$, $P(B) = 0.5$, $P(C) = 0.2$
   (c) $P(A) = 0.8$, $P(B) = 0.4$, $P(C) = 0.2$

3. (4 pts) Suppose that Ω is the set of all positive integers and let $\mathcal{C}$ be the collection of all finite subsets of Ω. Here a finite set means a set with finitely many element(s). Let
$$\mathcal{F} = \{A : A \in \mathcal{C} \text{ or } A^c \in \mathcal{C}\}.$$
   Determine whether $\mathcal{F}$ a σ-field on Ω and justify your answer.

4. (6 pts) Consider the experiment of rolling a die twice independently. Let X_i be the number obtained from the i-th rolling for $i = 1, 2$. Suppose that $P(X_i = k) = 1/6$ for $k \in \{1, 2, 3, 4, 5, 6\}$ for $i \in \{1, 2\}$. Let A_1 be the event that X_1 is an even number and A_2 be the event that X_2 is an odd number. Let A_3 be the event that $X_1 + X_2$ is an odd number.
   (a) Show that A_1 and A_3 are independent.
   (b) Show that A_2 and A_3 are independent.
   (c) Show that the three events A_1, A_2, A_3 are not independent.

5. (4 pts) Suppose that A_1, A_2, A_3, A_4 are events in a σ-field on Ω. Suppose that $P(A_i) = 0.1$ and $P(A_i \cap A_j) = 0.05$ for i, $j \in \{1, 2, 3, 4\}$. Find a lower bound and an upper bound for $P(A_1 \cup A_2 \cup A_3 \cup A_4)$. Be sure that the lower bound is greater than 0 and the upper bound is less than 1.

6. (6 pts) Suppose that P is a probability function defined on $\mathcal{F}$: a σ-field on Ω and A is an event in $\mathcal{F}$ such that $P(A) > 0$. For $B \in \mathcal{F}$, define $Q(B) = P(B|A)$. Show that Q is also a probability function defined on $\mathcal{F}$.

7. (6 pts) Given a sample space Ω, for $\mathcal{F}$ that is a collection of subsets of Ω, $\mathcal{F}$ is called a field on Ω if $\mathcal{F}$ satisfies the following three conditions.
   (a) $\emptyset \in \mathcal{F}$ and $\Omega \in \mathcal{F}$.
   (b) $A \in \mathcal{F}$ implies that $A^c \in \mathcal{F}$.
   (c) For each positive integer k, $A_1, \ldots, A_k \in \mathcal{F}$ implies that $\cup_{i=1}^{k} A_i \in \mathcal{F}$.

Show that if $\mathcal{F}$ is a field on Ω and $\mathcal{F}$ is a finite collection, then $\mathcal{F}$ is a σ-field on Ω.

8. (6 pts) Suppose that $\mathcal{F}$ is a σ-field on Ω and P is a probability function defined on $\mathcal{F}$. Suppose that A_1, A_2, …, are events in $\mathcal{F}$ such that $A_n \supset A_{n+1}$ for all n. Show that

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n). \tag{1}$$

   Note that we proved (1) in class for the case where $A_n \subset A_{n+1}$ for all n. you may use the result that was proved in class to solve this problem.

9. (6 pts) Suppose that F is the CDF of a random variable X. Show that

$$\lim_{x\to-\infty} F(x) = 0.$$

   Note that F is a bounded increasing function, so $\lim_{x\to-\infty} F(x)$ exists and you may evaluate the limit using $\lim_{n\to\infty} F(a_n)$ by choosing some sequence $\{a_n\}$ such that $\lim_{n\to\infty} a_n = -\infty$.

10. (12 pts) Consider the following function F:

$$F(x) = \begin{cases} 0.5 + 0.5x & \text{if } 0 < x < 1; \\ c_1 & \text{if } x < 0; \\ c_2 & \text{if } x > 1; \\ c_3 & \text{if } x = 0; \\ c_4 & \text{if } x = 1. \end{cases}$$

   where c_1, c_2, c_3, c_4 are constants. Suppose that F is the CDF of a random variable X.

   (a) (8 pts) Find c_1, c_2, c_3, c_4.

   (b) (4 pts) Explain why X is neither a discrete random variable nor a continuous random variable.

11. (8 pts) Suppose that X is a random variable with the CDF given in Problem 10.

   (a) Find $P(0 < X \le 1)$.

   (b) Find $P(X = 0)$.

   (c) Find $P(X = 1)$.

   (d) Find $P(0 \le X \le 1)$.

   If you choose not to turn in Problem 10, you may express your answers in terms of c_1, c_2, c_3, c_4.

12. (4 pts) Suppose that X is a discrete random variable with PMF p_X, which is given below:

$$p_X(x) = \begin{cases} 0.2 & \text{if } x = -1; \\ 0.4 & \text{if } x = 0; \\ c \cdot (0.5)^x & \text{if } x \in \{1, 2, 3, \ldots\}; \\ 0 & \text{otherwise,} \end{cases}$$

   where $c > 0$ is a constant.

(a) Find c.

(b) Find $P(X > 25)$.

13. (12 pts) Suppose that X is a discrete random variable with PMF p_X, which is given below:

$$p_X(x) = \begin{cases} 0.2 & \text{if } x = -1; \\ 0.5 & \text{if } x = 0; \\ 0.2 & \text{if } x = 1; \\ 0.1 & \text{if } x = 2; \\ 0 & \text{otherwise,} \end{cases}$$

(a) Find the CDF of X.

(b) Let $Y = (X-1)^2$, then Y is also a discrete random variable. Find the PMF of Y.

(c) Let F be the CDF of X and $Z = F(X)$. Find the PMF of Z.

14. (4 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

for $x \in (-\infty, \infty)$. Let $Y = 1 + 2X$. Find a PDF of Y.

15. (4 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = \begin{cases} 0 & \text{if } x \le 0; \\ 2xe^{-x^2} & \text{if } x > 0, \end{cases}$$

Let F be the CDF of X. Find a PDF of $Y = F(X)$.

16. (4 pts) Suppose that X has PDF f_X, where

$$f_X(x) = \begin{cases} (2+x)/3 & \text{if } -2 < x \le 0; \\ 2(1-x)/3 & \text{if } 0 < x < 1; \\ 0 & \text{if } x \notin (-2, 1), \end{cases}$$

Let $Y = X^2$. Find the PDF of Y.

17. (4 pts) Consider the X in Problem 15. Find the median and the interquartile range of the distribution of X.

18. (4 pts) Suppose that X is a discrete random variable with PMF p_X, where

$$p_X(k) = \begin{cases} C_k^3 (0.6)^k (0.4)^{3-k} & \text{if } k \in \{0, 1, 2, 3\}; \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X^2)$.

19. (4 pts) Suppose that $\lambda > 0$ is a constant and X is a discrete random variable with PMF p_X, where

$$p_X(k) = \begin{cases} e^{-\lambda}\lambda^k / k! & \text{if } k \in \{0, 1, 2, \ldots\}; \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X(X-1))$.

20. (4 pts) Suppose that X is a random variable with PDF f_X and $f_X(2-x) = f_X(x)$ for $x \in (-\infty, \infty)$. Show that the median of the distribution of X is 1.

21. (4 pts) Suppose that X is a random variable with PDF f_X, where f_X is a continuous function on $(-\infty, \infty)$ and $f_X(x) > 0$ for $x \in (-\infty, \infty)$. Let U be a random variable with PDF f_U, where

$$f_U(u) = \begin{cases} 1 & \text{if } u \in (0,1); \\ 0 & \text{otherwise.} \end{cases}$$

Let F be the CDF of X and $Y = F^{-1}(U)$. Show that f_X is a PDF of Y.

22. (4 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = \begin{cases} 1/(2x^2) & \text{if } |x| > 1; \\ 0 & \text{if } |x| \le 1, \end{cases}$$

Find $E(X)$.

23. (4 pts) Consider the X in Problem 22. Find the PDF of $Y = 1/X$.

24. (4 pts) Consider the X in Problem 22. Find $E(1/X)$.

25. (4 pts) Suppose that X is a random variable with finite mean μ and standard deviation $\sigma > 0$. Let $Y = (X - \mu)/\sigma$. Find $E(Y)$ and $Var(Y)$ when $\mu = 3$ and $\sigma = 2$.

26. (4 pts) Suppose that Z is a discrete random variable with n possible values $z_1, \ldots, z_n$. Suppose that h_1 and h_2 are two real valued functions defined on $\{z_1, \ldots, z_n\}$. Show that

$$E(h_1(Z) + h_2(Z)) = E(h_1(Z)) + E(h_2(Z))$$

without using the property that

$$E(X + Y) = E(X) + E(Y)$$

for random variables X and Y such that $E(|X|)$ and $E(|Y|)$ are finite.

27. (4 pts) Suppose that X is a discrete random variable with n possible values $x_1, \ldots, x_n$ and m and M are two constants such that $m \le x_i \le M$ for $i \in \{1, \ldots, n\}$. Show that

$$m \le E(X) \le M. \tag{2}$$

   - Remark. In general, for a random variable X such that $P(m \le X \le M) = 1$ for some constants m and M, (2) holds.

28. (4 pts) Suppose that X is a random variable with MGF M_X, where

$$M_X(t) = 0.6 + 0.4e^{2t}$$

for $t \in (-\infty, \infty)$. Find $E(X^k)$ for $k \in \{1, 2, 3, 4\}$.

29. (4 pts) Suppose that Y is a discrete random variable with PMF p_Y, where

$$p_Y(y) = \begin{cases} 0.6 & \text{if } y = 0; \\ 0.4 & \text{if } y = 2; \\ 0 & \text{otherwise.} \end{cases}$$

Show that Y and the X in Problem 28 have the same CDF by verifying that the two random variables have the same MGF.

30. (10 pts) Consider the random variable X in Problem 19.

    (a) (4 pts) Find the MGF of X.

    (b) (4 pts) Find $E(X)$ and $E(X^2)$ using the MGF of X.

    (c) (2 pts) Find $E(X(X-1))$ using the result in Part (b).

31. (6 pts) Suppose that (X, Y) is a vector of discrete random variables with joint PMF $p_{X,Y}$, where

$$p_{X,Y}(x, y) = \begin{cases} 0.5 & \text{if } (x, y) = (1, 2); \\ 0.1 & \text{if } (x, y) = (3, 2); \\ 0.3 & \text{if } (x, y) = (3, 6); \\ 0.1 & \text{if } (x, y) = (3, 7); \\ 0 & \text{otherwise.} \end{cases}$$

    (a) Find $P((X, Y) \in (-\infty, 3] \times (-\infty, 2])$.

    (b) Find $P((X + Y) \leq 9)$.

    (c) Find the (marginal) PMF of Y.

32. (10 pts) Suppose that (X, Y) has joint PDF of $f_{X,Y}$, where

$$f_{X,Y}(x, y) = \begin{cases} cx & \text{if } (x, y) \in (0, 1) \times (0, 1); \\ 0 & \text{otherwise} \end{cases}$$

and $c > 0$ is a constant.

    (a) (3 pts) Find c.

    (b) (3 pts) Find $P((X + 2Y) \leq 1)$.

    (c) (4 pts) Find a PDF of Y.

33. (10 pts) Suppose that (X, Y) has joint CDF $F_{X,Y}$, where

$$F_{X,Y}(x, y) = \begin{cases} 0.5G(x)G(y) + 0.5(1 - e^{-x})(1 - e^{-y}) & \text{if } x \geq 0 \text{ and } y \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

and

$$G(x) = \begin{cases} 0 & \text{if } x < 0; \\ x & \text{if } 0 \leq x < 1; \\ 1 & \text{if } x \geq 1. \end{cases}$$

    (a) Find $P(0 < X \leq 1 \text{ and } 1 < Y \leq 2)$.

    (b) Find the CDF of X.

34. (4 pts) Suppose that (X, Y, Z) is a vector of random variables with joint CDF F and a, b, c, d, e, f are constants such that $a < b$, $c < d$ and $e < f$. Express

$$P((X, Y, Z) \in (a, b] \times (c, d] \times (e, f])$$

in terms of F and a, b, c, d, e, f.

35. (10 pts) Suppose that (X, Y) has joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = \begin{cases} ce^{-(x^2+y^2)/2} & \text{if } (x, y) \in (0, \infty) \times (0, \infty); \\ 0 & \text{otherwise,} \end{cases}$$

and c is a positive constant. Let $U = \sqrt{X^2 + Y^2}$ and $V = \tan^{-1}(Y/X)$. Note that for $z \in (-\infty, \infty)$, $\tan^{-1}(z)$ is defined to be the value θ such that $\tan(\theta) = z$ and $\theta \in (-\pi/2, \pi/2)$.

(a) (4 points) Find the joint PDF of (U, V). Leave the constant c in your answer.

(b) (4 points) Find the PDF of V. Leave the constant c in your answer.

(c) (2 points) Find c using your answer in Part (b) and the fact that the integral of the PDF of V over $(-\infty, \infty)$ is 1.

Remark. Let $I = \int_0^\infty e^{-x^2/2} dx$, then

$$1 = \int_{R^2} f_{X,Y}(x, y) d(x, y) = cI^2,$$

so we have $I = 1/\sqrt{c}$ and $\int_{-\infty}^\infty e^{-x^2/2} dx = 2I = 2/\sqrt{c}$.

36. (10 points) Define

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

for $a > 0$. Suppose that α and β are two positive constants and (X, Y) is a random vector with joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} y^{\beta-1} e^{-x-y} & \text{if } (x, y) \in (0, \infty) \times (0, \infty); \\ 0 & \text{otherwise.} \end{cases}$$

(a) (4 pts) Find the joint MGF of (X, Y).

(b) (2 pts) Find the MGF of X.

(c) (4 pts) Find $E(XY)$ and $E(X)$.

37. (10 pts) Consider the (X, Y) in Problem 36. Find a PDF of $U = X/(X + Y)$.

Hint: take $V = v(X, Y)$ for some function v and then find $f_{U,V}$ and f_V.

38. (10 pts) Suppose that (X, Y) is a vector of two discrete random variables with joint PMF $p_{X,Y}$, where

$$p_{X,Y}(x, y) = \begin{cases} 0.2 & \text{if } (x, y) = (-1, 2); \\ 0.6 & \text{if } (x, y) = (2, -3); \\ 0.2 & \text{if } (x, y) = (4, -3); \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 pts) Find the PMF of $1/(X + Y)$.

(b) (5 pts) Find $E(1/X + 1/Y)$.

39. (10 pts) Suppose that λ and μ are positive constants and (X, Y) is a vector of two discrete random variables with joint PMF $p_{X,Y}$, where

$$p_{X,Y}(x, y) = \begin{cases} e^{-\lambda-\mu} \lambda^x \mu^y / (x!y!) & \text{if } x, y \in \{0, 1, 2, \ldots\}; \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 pts) Find the PMF of $X+Y+1$.

(b) (5 pts) Find $E(X+Y+1)$.

40. (12 pts) Suppose that (X,Y) has a joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } (x,y) \in \{(x,y) : -2 < x+2y < 2 \text{ and } -2 < x-2y < 2\}; \\ 0 & \text{otherwise,} \end{cases}$$

and c is the constant such that $\int_{R^2} f_{X,Y}(x,y)d(x,y) = 1$.

(a) (4 pts) Find $E(X|Y)$. You may leave c in your answer.

(b) (4 pts) Find $Var(X|Y)$. You may leave c in your answer.

(c) (4 pts) Determine whether $X+2Y$ and $X-2Y$ are independent. Justify your answer.

41. (14 pts) Suppose that (X,Y) is a vector of two discrete random variables with joint PMF $p_{X,Y}$, where

$$p_{X,Y}(x,y) = \begin{cases} 0.5 & \text{if } (x,y) = (1,2); \\ 0.4 & \text{if } (x,y) = (0,-3); \\ 0.1 & \text{if } (x,y) = (1,-3); \\ 0 & \text{otherwise.} \end{cases}$$

(a) (4 pts) Find $E(X|Y=-3)$ and $Var(X|Y=-3)$.

(b) (4 pts) Find $E(X|Y)$ and $Var(X|Y)$.

(c) (4 pts) Find $E(Var(X|Y))$, $Var(E(X|Y))$ and $Var(X)$. Check whether

$$Var(X) = E(Var(X|Y)) + Var(E(X|Y))$$

based on your answers.

(d) (2 pts) Determine whether X and Y are independent. Justify your answer.

42. (12 pts) For $a > 0$, define

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$$

and

$$f_a(x) = \begin{cases} \frac{1}{\Gamma(a)}x^{a-1}e^{-x} & \text{if } x > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that X and Y are two random variables with PDFs f_α and f_β respectively, where α and β are positive constants. Suppose that X and Y are independent. To solve this problem, you may use the joint MGF of (X,Y) and the MGF of X given in the solution to Problem 36.

(a) (4 pts) Find the MGF of $X+Y$.

(b) (2 pts) Find a PDF of $X+Y$.

(c) (4 pts) Find the joint MGF of $(X+Y, X-Y)$.

(d) (2 pts) Determine whether $X+Y$ and $X-Y$ are independent. Justify your anser.

43. (4 pts) Suppose that (X, Y) has joint MGF $M_{X,Y}$ and $M_{X,Y}(t_1, t_2) = g(t_1)h(t_2)$ is finite for $|t_1| < \delta$, $|t_2| < \delta$, where δ is a positive constant. Suppose that $M_{X,Y}(t_1, t_2) = g(t_1)h(t_2)$ for $|t_1| < \delta$, $|t_2| < \delta$, where g and h are two functions. Can we deduce that X and Y are independent? Justify your answer.

44. (6 pts) Suppose that $X \sim N(\mu, \sigma^2)$, where $\mu \in (-\infty, \infty)$ and $\sigma > 0$. Let $Y = a + bX$, where a and b are constants and $b \neq 0$. Show that $Y \sim N(E(Y), Var(Y))$. You may use the MGF of a normal distribution or the PDF of a normal distribution given in Example 4 in the handout "Independent random variables".

45. (4 pts) Suppose that Z_1, Z_2, Z_3 are IID random variables and $Z_1 \sim N(0, 1)$. Let

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}.$$

Find a PDF of (Y_1, Y_2, Y_3).

46. (4 pts) Suppose that Z_1, …, Z_n are IID random variables and $Z_1 \sim N(0, 1)$. Suppose that A is an $n \times n$ matrix of constants whose inverse matrix exists and μ_1, …, μ_n are constants. Let

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = A \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}.$$

Find a PDF of $(Y_1, \ldots, Y_n)$ that is determined by AA^T and μ_1, …, μ_n. You may use results from linear algebra such as $\det(B^T) = \det(B)$ and

$$\det(BC) = \det(B) \cdot \det(C)$$

for square matrices B and C. Here $\det(A)$ denotes the determinant of a square matrix A.

47. (4 pts) Suppose that X_1, …, X_n are IID and $X_1 \sim N(\mu, \sigma^2)$. Let $\bar{X} = (\sum_{i=1}^{n} X_i)/n$ and $Y = (X_1 - \bar{X}, \ldots, X_n - \bar{X})$. Show that $\bar{X} - \mu$ and Y are independent.

48. (6 pts) Suppose that $Z \sim N(0, 1)$ and let $Y = Z^2/2$. Find the MGF of Y and find the constant a such that the function f_a in Problem 42 is a PDF of Y. You may use the results in the solutions to Problems 36 and 42.

49. (6 pts) Suppose that (X, Y) is a vector of two discrete random variables with joint PMF $p_{X,Y}$, where

$$p_{X,Y}(x, y) = \begin{cases} 0.5 & \text{if } (x, y) = (1, 2); \\ 0.4 & \text{if } (x, y) = (0, a); \\ 0.1 & \text{if } (x, y) = (-1, -2); \\ 0 & \text{otherwise,} \end{cases}$$

and a is a constant.

(a) (4 pts) Express $Corr(X, Y)$ as function of a.

(b) (2 pts) Find all a's such that $|Corr(X, Y)| = 1$.

50. (4 pts) Suppose that X is a random variable and $Var(X) > 0$. For constants a and b such that $b > 0$, let $Y = a + bX$. Show that $Var(Y) > 0$ and $Corr(X, Y) = 1$.

51. (16 pts) Suppose that (X, Y, Z) has a joint PDF $f_{X,Y,Z}$, where for $(x, y, z) \in R^3$,

$$f_{X,Y,Z}(x, y, z) = \begin{cases} e^{-(x^2+4xy+5y^2+5)} ze^{-z}/\pi & \text{if } z > 0; \\ 0 & \text{if } z \leq 0. \end{cases}$$

(a) (2 pts) Find a joint PDF of (X, Y).

(b) (2 pts) Find a version of the conditional PDF of Z given $(X, Y) = (x, y)$ for all $(x, y) \in R^2$.

(c) (2 pts) Find $E(Z|X, Y)$.

(d) (4 pts) Find a version of the conditional PDF of X given $Y = y$ for all $y \in (-\infty, \infty)$.

(e) (4 pts) Find $Cov(X, Y)$.

(f) (2 pts) Determine whether X and Y independent and justify your answer.

52. (4 pts) Suppose that (X, Y) has a joint PDF and $f_{X|Y=y}$ is a version of the conditional PDF of X given $Y = y$ for $y \in (-\infty, \infty)$. Then for g such that $E(g(X, Y))$ is finite, $E(g(X, Y)|Y)$ can be obtained using

$$E(g(X, Y)|Y = y) = \int g(x, y) f_{X|Y=y}(x) dx. \tag{3}$$

for all y. Use (3) to show that $E(XY|Y) = YE(X|Y)$ when $E(XY)$ is finite.

53. (4 pts) For $y \in (-\infty, \infty)$, define a function g_y as follows:

$$g_y(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-y)^2/2}$$

for $x \in (-\infty, \infty)$. Suppose that X and Y are random variables such that $Y \sim N(0, 1)$ and a version of the conditional PDF of X given $Y = y$ is g_y for all $y \in (-\infty, \infty)$. Find a version of the conditional PDF of Y given $X = x$ for all $x \in (-\infty, \infty)$.

54. (4 pts) Suppose that $Var(X) = 1$, $Var(Y) = 0.25$, and $Cov(X, Y) = 0.5$.

(a) (2 pts) Find $Corr(X, Y)$.

(b) (2 pts) Find $Var(X - 2Y)$.

55. (6 pts) Suppose that $\boldsymbol{X} = (X_1, \ldots, X_m)^T$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)^T$ are two random vectors. Define the covariance matrix for the pair $(\boldsymbol{X}, \boldsymbol{Y})$ as

$$Cov(\boldsymbol{X}, \boldsymbol{Y}) = E((\boldsymbol{X} - \boldsymbol{\mu}_1)(\boldsymbol{Y} - \boldsymbol{\mu}_2)^T),$$

where $\boldsymbol{\mu}_1 = E(\boldsymbol{X})$ and $\boldsymbol{\mu}_2 = E(\boldsymbol{Y})$.

(a) (3 pts) Suppose that $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant column vectors of length m and n respectively. Show that

$$Cov(\boldsymbol{X} + \boldsymbol{a}, \boldsymbol{Y} + \boldsymbol{b}) = Cov(\boldsymbol{X}, \boldsymbol{Y}).$$

(b) (3 pts) Suppose that A and B are constant matrice with numbers of columns m and n respectively. Show that

$$Cov(A\boldsymbol{X}, B\boldsymbol{Y}) = ACov(\boldsymbol{X}, \boldsymbol{Y})B^T.$$

Note.

- The (i,j)-th element of $Cov(\boldsymbol{X}, \boldsymbol{Y})$ is $Cov(X_i, Y_j)$.
- $Cov(\boldsymbol{X}, \boldsymbol{X})$ is the covariance matrix of $\boldsymbol{X}$ defined in the handout "Multivariate normal distributions".
- From the results of this problem, we have that the covariance matrix of $A\boldsymbol{X} + \boldsymbol{b}$ is

$$Cov(A\boldsymbol{X} + \boldsymbol{b}, A\boldsymbol{X} + \boldsymbol{b}) = Cov(A\boldsymbol{X}, A\boldsymbol{X}) = ACov(\boldsymbol{X}, \boldsymbol{X})A^T. \quad (4)$$

This result in (4) is also given in the handout "Multivariate normal distributions" (before Example 3).

56. (6 pts) Suppose that $\boldsymbol{X} = (X_1, X_2, X_3, X_4)^T$ is a random vector with $E(\boldsymbol{X}) = (0, 0, 0, 0)^T$ and covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(a) (3 pts) Find $Var(X_1 + X_2 + X_3)$ by finding a matrix A such that $AX = (X_1 + X_2 + X_3)$ and then applying (4).

(b) (3 pts) Find the best linear predictor of X_1 based on X_2.

57. (6 pts) Consider the (X_1, X_2, X_3, X_4) in Problem 56. Define the best linear predictor X_1 based on X_2 and X_3 to be the linear combination $a_0 + b_0 X_2 + c_0 X_3$ such that

$$E(X_1 - (a + bX_2 + cX_3))^2$$

is minimized at $(a, b, c) = (a_0, b_0, c_0)$. Find the best linear predictor X_1 based on X_2 and X_3.

58. (9 pts) Consider the (X_1, X_2, X_3, X_4) in Problem 56. Suppose that the distribution of (X_1, X_2, X_3, X_4) is a multivariate normal distribution.

(a) (6 pts) Show that (X_1, X_2) and (X_3, X_4) are independent by verifying that

$$M_{X_1,X_2,X_3,X_4}(t_1, t_2, t_3, t_4) = M_{X_1,X_2}(t_1, t_2) M_{X_3,X_4}(t_3, t_4)$$

for $(t_1, t_2, t_3, t_4) \in R^4$. Here M_{X_1,X_2,X_3,X_4}, M_{X_1,X_2} and M_{X_3,X_4} are the moment generating functions of (X_1, X_2, X_3, X_4), (X_1, X_2) and (X_3, X_4) respectively.

(b) (3 pts) Let $\bar{X} = \sum_{i=1}^{4} X_i/4$. Find the distribution of $\bar{X}$.

59. (6 pts) Suppose that $\boldsymbol{X} = (X_1, \ldots, X_m)^T$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_n)^T$ are two random vectors and the distribution of $(\boldsymbol{X}^T, \boldsymbol{Y}^T)$ is a multivariate normal distribution. Suppose that $Cov(X_i, Y_j) = 0$ for every (i, j) such that $1 \le i \le m$ and $1 \le j \le n$. That is, $Cov(\boldsymbol{X}, \boldsymbol{Y})$ is a matrix of 0's. Show that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent.

60. (8 pts) Suppose that X amd ε are random variables such that $Var(X) > 0$, $Var(\varepsilon) > 0$, $Cov(X, \varepsilon) = 0$ and $E(\varepsilon) = 0$. Let $Y = 1 + 2X + \varepsilon$.

   (a) (4 pts) Show that the best linear predictor of Y based on X is $1+2X$.

   (b) (4 pts) For $\mu \in (-\infty, \infty)$ and $\sigma > 0$, define the function $f_{\mu,\sigma}$ as follows:

$$f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} \text{ for } x \in (-\infty, \infty). \tag{5}$$

   Suppose that the distribution(X, ε) is a multivariate normal distribution and $Var(\varepsilon) > 0$. Show that a version of the conditional PDF of Y given $X = x$ is $f_{\mu,\sigma}$ with $\mu = 1+2x$ and $\sigma^2 = E(Y - (1+2X))^2$ for $x \in (-\infty, \infty)$.

61. (4 pts) Suppose that $(X_{i,1}, X_{i,2}, Y_i)$: $i = 1, \ldots, n$ are n vectors of observations for variables X_1, X_2 and Y. A way to investigate the relation between Y and (X_1, X_2) is to find constants a, b_1, b_2 such that $Y_i \approx a + b_1 X_{i,1} + b_2 X_{i,2}$ using least square estimation. That is, find a, b_1, and b_2 so that

$$RSS(a, b_1, b_2) = \sum_{i=1}^{n} (Y_i - (a + b_1 X_{i,1} + b_2 X_{i,2}))^2$$

is minimized. Suppose that $RSS(a, b_1, b_2)$ is minimized at $(a, b_1, b_2) = (\hat{a}, \hat{b}_1, \hat{b}_2)$. Let $\bar{X}_j = \sum_{i=1}^n X_{i,j}/n$ for $j = 1, 2$, $\bar{Y} = \sum_{i=1}^n Y_i/n$ and

$$S_{X_1,X_2} = \begin{pmatrix} \sum_{i=1}^n (X_{i,1} - \bar{X}_1)^2 & \sum_{i=1}^n (X_{i,1} - \bar{X}_1)(X_{i,2} - \bar{X}_2) \\ \sum_{i=1}^n (X_{i,1} - \bar{X}_1)(X_{i,2} - \bar{X}_2) & \sum_{i=1}^n (X_{i,2} - \bar{X}_2)^2 \end{pmatrix}.$$

Show that

$$S_{X_1,X_2} \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n (X_{i,1} - \bar{X}_1)(Y_i - \bar{Y}) \\ \sum_{i=1}^n (X_{i,2} - \bar{X}_2)(Y_i - \bar{Y}) \end{pmatrix},$$

and

$$\bar{Y} = \hat{a} + \hat{b}_1 \bar{X}_1 + \hat{b}_2 \bar{X}_2.$$

Hint: you may consider a random vector (W, V_1, V_2) so that $P((W, V_1, V_2) = (Y_i, X_{i,1}, X_{i,2})) = 1/n$ for $i = 1, \ldots, n$ (treating $(Y_i, X_{i,1}, X_{i,2})$ as non-random), then $E(W - (a + b_1 V_1 + b_2 V_2))^2 = RSS(a, b_1, b_2)/n$ and then apply the results in Equations (3) and (4) in the handout “Multivariate normal distribution”.

62. (10 pts) Suppose that $\boldsymbol{X} = (X_1, X_2, X_3, X_4)^T$ is a random vector with $E(\boldsymbol{X}) = (1, 2, 3, 4)^T$ and covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

   (a) (4 pts) Find the best linear predictor of X_4 based on X_1 and X_3.

   (b) (6 pts) Suppose that $\boldsymbol{X} \sim N(E(\boldsymbol{X}), \Sigma)$. Find a constant b such that $X_3 - bX_2$ and X_2 are independent. In addition, find $E(X_3|X_2)$ and $Var(X_3|X_2)$.

63. (8 pts) Suppose that $\boldsymbol{X} = (X_1, X_2, X_3, X_4, X_5)^T$ is a random vector with $E(\boldsymbol{X}) = (1, 2, 3, 4, 5)^T$ and covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 2 & 1 & 0.5 & 0 & 0 \\ 1 & 2 & 1 & 0.5 & 0 \\ 0.5 & 1 & 2 & 0 & 0 \\ 0 & 0.5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Suppose that $\boldsymbol{X} \sim N(E(\boldsymbol{X}), \Sigma)$.

(a) (6 pts) Find a version of the conditional PDF of (X_1, X_2) given $(X_3, X_4, X_5) = (x_3, x_4, x_5)$ for $(x_3, x_4, x_5) \in R^3$.

(b) (2 pts) Find a version of the conditional PDF of X_5 given $(X_1, X_2, X_3, X_4) = (x_1, x_2, x_3, x_4)$ for $(x_1, x_2, x_3, x_4) \in R^4$.

64. (6 pts) Consider the random vector $\boldsymbol{X} = (X_1, X_2, X_3, X_4, X_5)^T$ in Problem 63. Let $\boldsymbol{X}_0 = (X_2, X_3, X_4)^T$ Find a 3×3 invertible matrix A such that all the compoments in $A\boldsymbol{X}_0$ are independent.

65. (4 pts) Suppose that X is a nonnegative random variable and $E(X)$ is finite. Show that for a positive constant c,

$$P(X \geq c) \leq \frac{E(X)}{c}. \tag{6}$$

You may use the following result in your proof.

Fact 1 *Suppose that X_1 and X_2 are two random variables such that $X_1 \leq X_2$ and $E(X_1)$ and $E(X_2)$ are finite. Then $E(X_1) \leq E(X_2)$.*

Remark. (6) is known as the Markov inequality.

66. (4 pts) Suppose that n_1, n_2 are positive integers and p is a constant in $(0, 1)$. Suppose that X and Y are two independent random variables such that $X \sim b(n_1, p)$ and $Y \sim b(n_2, p)$. Show that $X + Y \sim b(n_1 + n_2, p)$.

Hint: compute the MGF of $X + Y$ and compare it with the MGF of $b(n_1 + n_2, p)$.

67. (4 pts) Suppose that $\alpha > 0$ and $\beta > 0$. Define

$$f_{\alpha,\beta}(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} & \text{if } x > 0; \\ 0 & \text{if } x \leq 0, \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Then the distribution with PDF $f_{\alpha,\beta}$ is the gamma distribution $\Gamma(\alpha, \beta)$. Let $M_{\alpha,\beta}$ be the MGF of $\Gamma(\alpha, \beta)$, then we have shown in class that

$$M_{\alpha,\beta}(t) = (1 - \beta t)^{-\alpha}$$

for $t < 1/\beta$. Suppose that λ is a positive constant, T_1, ..., T_k are IID random variables, and $T_1 \sim \Gamma(1, 1/\lambda)$. Show that $\sum_{i=1}^k T_i \sim \Gamma(k, 1/\lambda)$. Do not use Fact 3 in the handout "Some special distributions" to solve this problem.

68. (16 pts) For $(y_1, y_2, y_3) \in R^3$, define

$$g_{y_2,y_3}(y_1) = \frac{1}{\sqrt{2\pi}} e^{-(y_1-y_2-y_3)^2/2}$$

and

$$h(y_2, y_3) = \left(\frac{\sqrt{3}}{2\pi}\right) e^{-(2y_2^2+2y_3^2+2y_2y_3)/2}.$$

Suppose that (Y_1, Y_2, Y_3) is a random vector, h is a PDF of (Y_2, Y_3) and g_{y_2,y_3} is a version of the conditional PDF of Y_1 given $(Y_2, Y_3) = (y_2, y_3)$ for $(y_2, y_3) \in R^2$. We have verified the following results in class:

(i) h is a PDF of $N(\boldsymbol{\mu}_1, \Sigma_1)$ with $\boldsymbol{\mu}_1 = (0, 0)^T$ and

$$\Sigma_1 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

(ii) $E(Y_1|Y_2, Y_3) = Y_2 + Y_3$.

(iii) $Cov(Y_1, Y_2) = 1/3$.

(a) (4 pts) Find $Cov(Y_1, Y_3)$.

(b) (4 pts) Show that $E(Y_1^2|Y_2, Y_3) = 1 + (Y_2 + Y_3)^2$.

(c) (4 pts) Let Σ be the covariance matrix of $(Y_1, Y_2, Y_3)^T$. Find Σ.

(d) (4 pts) Let $\boldsymbol{\mu} = (0, 0, 0)^T$. Show that $(Y_1, Y_2, Y_3)^T \sim N(\boldsymbol{\mu}, \Sigma)$, where Σ is the covariance matrix of $(Y_1, Y_2, Y_3)^T$ found in Part (c).