Homework Problems

- Note. Always show your work in your homework solutions to receive full points unless it is stated otherwise.

1. (5 pts) Suppose that P is a probability function defined on a σ-field $\mathcal{F}$. For an event $B \in \mathcal{F}$ such that $P(B) > 0$, define a function Q on $\mathcal{F}$ by

$$Q(A) = \frac{P(A \cap B)}{P(B)}$$

for $A \in \mathcal{F}$. Verify that Q is a probabilty function on $\mathcal{F}$.

2. (5 pts) Suppose that P is a probability function defined on a σ-field $\mathcal{F}$, and $\{A_n\}_{n=1}^{\infty}$ is a sequence of events in $\mathcal{F}$ such that $A_n \supset A_{n+1}$ for all n. Show that

$$\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n).$$

3. (5 pts) Suppose that $\mathcal{F}$ is a σ-field on $\Omega = (-\infty, \infty)$ such that all open intervals in $(-\infty, \infty)$ are in $\mathcal{F}$. Suppose that P is a probability function on $\mathcal{F}$ such that for $n \in \{1, 2, \dots, \}$,

$$P\left(\left(-\frac{1}{n}, \frac{1}{n}\right)\right) = 0.4 + \frac{0.6}{n}.$$

Find $P(\{0\})$.

4. (5 pts) Suppose that $\mathcal{F}$ is a σ-field on $\Omega = \{1, 2, 3, 4, 5\}$. Let $A = \{1, 2, 4, 5\}$, $B = \{1, 2, 4\}$ and $C = \{1, 4, 5\}$. Suppose that A, B, C are in $\mathcal{F}$ and P is a probability function defined on $\mathcal{F}$. Express $P(\{5\})$, $P(\{1, 4\})$, $P(\{2\})$ and $P(\{3\})$ in terms of $P(A)$, $P(B)$ and $P(C)$. Explain why we cannot have

$$(P(A), P(B), P(C)) = (0.5, 0.3, 0.1).$$

5. (4 pts) Suppose that A_1, A_2, A_3, A_4 are events in a σ-field on Ω. Suppose that $P(A_i) = 0.1$ and $P(A_i \cap A_j) = 0.05$ for $i,\ j \in \{1, 2, 3, 4\}$. Find a lower bound and an upper bound for $P(A_1 \cup A_2 \cup A_3 \cup A_4)$. Be sure that the lower bound is greater than 0 and the upper bound is less than 1.

6. (4 pts) Suppose that $\mathcal{F}$ is a σ-field on a space Ω. Suppose that $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{F}$. Show that $\cap_{n=1}^{\infty} A_n \in \mathcal{F}$.

7. (8 pts) Suppose that $\Omega = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Let $\mathcal{C} = \{\emptyset, \Omega, A, B\}$. Suppose that $\mathcal{F}$ is the smallest σ-field on Ω and $\mathcal{C} \subset \mathcal{F}$. List all sets that should be included in $\mathcal{F}$ and explain why they should be in $\mathcal{F}$. You do not have to verify that the collection of sets in your list is a σ-field, but be sure it is.

8. (4 pts) Consider the space Ω and the σ-field $\mathcal{F}$ in Problem (7). For $w \in \Omega$, define

$$X(w) = \begin{cases} 10 & \text{if } w = 1; \\ 20 & \text{if } w \neq 1. \end{cases}$$

Explain why X is not a measurable function from $(\Omega, \mathcal{F})$ to $(R, \mathcal{B}(R))$.

9. (4 pts) Suppose that X is a random variable with CDF F, where

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ 0.5 + 0.5x & \text{if } 0 \leq x < 1; \\ 1 & \text{if } x \geq 1; \end{cases}$$

Find $P(X = a)$ for every $a \in R$.

10. (4 pts) Consider the experiment of tossing a coin twice independently. Let X be the total number of heads obtained in this experiment. Specify a probability space $(\Omega, \mathcal{F}, P)$ so that X is a random variable on the probability space. You may write down your answer directly without justification.

11. (6 pts) Suppose that F is the CDF of a random variable X. Show that $\lim_{x\to-\infty} F(x) = 0$. You may use the fact that $\lim_{x\to-\infty} F(x)$ exists without proving it.

12. (16 pts) Consider the following function F:

$$F(x) = \begin{cases} c_1 & \text{if } x < 0; \\ 0.5 + 0.5x & \text{if } 0 < x < 1; \\ c_2 & \text{if } x > 1; \\ c_3 & \text{if } x = 0; \\ c_4 & \text{if } x = 1, \end{cases}$$

where c_1, c_2, c_3, c_4 are constants. Suppose that F is the CDF of some random variable X.

(a) (8 pts) Find c_1, c_2, c_3, c_4.

(b) (4 pts) Explain why X is not a discrete random variable.

(c) (4 pts) Find $P(0 \leq X \leq 1)$.

13. (4 pts) Suppose that X is a discrete random variable with PMF p_X, which is given below:

$$p_X(x) = \begin{cases} 0.2 & \text{if } x = -1; \\ 0.4 & \text{if } x = 0; \\ c \cdot (0.5)^x & \text{if } x \in \{1, 2, 3 \ldots, \}; \\ 0 & \text{otherwise,} \end{cases}$$

where $c > 0$ is a constant.

(a) Find c.

(b) Find $P(X > 25)$.

14. (6 pts) Suppose that X is a random variable with CDF F, where

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ 1 - (0.5)^{(k+1)} & \text{if } k \leq x < (k+1) \text{ for } k \in \{0, 1, 2\}; \\ 1 & \text{if } x \geq 3. \end{cases}$$

Let $A = (-0.5, 1.5) \cup (2.2, 3.5) \cup (4, 5)$. Find $P(X \in A)$.

15. (6 pts) Suppose that X is a random variable with CDF F, where

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ 1 - e^{-2x} & \text{if } x \geq 0. \end{cases}$$

Find a PDF of X and find $P(X > 2)$.

16. (6 pts) Suppose that X is a random variable with PDF f_X. Let $S_X = \{x : f_X(x) > 0\}$. Suppose that S_X is an open interval and f_X is continuous on S_X. Let F be the CDF of X. Suppose that $F' > 0$ and is continuous on S_X. Then it can be shown that inverse function F^{-1} is defined on $(0, 1)$ (but you don't have to prove this result). Suppose that U is a random variable with PDF f_U, where

$$f_U(x) = \begin{cases} 1 & \text{if } x \in (0,1); \\ 0 & \text{otherwise.} \end{cases}$$

Show that f_X is a PDF of $F^{-1}(U)$.

Note:

- The disitrbution of U is called the uniform distribution on $(0, 1)$, denoted by $U(0, 1)$.
- The result that X and $F^{-1}(U)$ have the same distribution can be established under weaker conditions.
- From now on, we will use the notation I_A to denote the indicator function of A for a given set A, which is defined by

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$$

For instance, the function f_U is the indicator function $I_{(0,1)}$.

17. (6 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = 2xe^{-x^2} I_{(0,\infty)}(x) \text{ for } x \in R.$$

Find a PDF of $Y = \sqrt{X}$.

18. (6 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = |x| I_{(-1,0)}(x) + 0.5e^{-x} I_{(0,\infty)}(x)$$

Find a PDF of $Y = X^2$.

19. (6 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = e^{-x} I_{(0,\infty)}(x) \text{ for } x \in R.$$

Suppose that

$$Y = \begin{cases} X & \text{if } X \le 0.5; \\ 0.5 & \text{if } X > 0.5. \end{cases}$$

Find the CDF of Y and explain why Y does not have a PDF.

20. (8 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = 2xe^{-x^2} I_{(0,\infty)}(x) \text{ for } x \in R.$$

(a) (4 pts) Find the CDF of X.

(b) (4 pts) Find the median and the IQR of the distribution of X.

21. (4 pts) Suppose that X is a random variable such that both $E(X^2)$ and $E(|X|)$ are finite. Verify that $Var(X) = E(X^2) - (E(X))^2$ using Properties (i)–(iii) listed in Page 6 of the handout "Quantile and expectation".

22. (4 pts) Suppose that X is a random variable with finite expectation μ and standard deviation $\sigma > 0$ ($\sigma = \sqrt{Var(X)}$). Let $Y = (X - \mu)/\sigma$. Find $E(Y)$ and $Var(Y)$ with $\mu = 1.5$ and $\sigma = 1.2$.

23. (4 pts) Suppose that X is a discrete random variable with PMF p_X, where for $x \in R$,

$$p_X(x) = \begin{cases} C_x^3(0.6)^x(0.4)^{3-x} & \text{if } x \in \{0, 1, 2, 3\}; \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X^2)$.

24. (8 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = \frac{1}{2x^2} \cdot I_{(-\infty,-1)\cup(1,\infty)}(x)$$

for $x \in R$.

(a) (4 ps) Find a PDF of $1/X$.

(b) (4 ps) Find $E(1/X)$.

25. (8 pts) Suppose that X and Y are discrete random variables, and

$$P((X,Y) = (x,y)) = \begin{cases} 0.5 & \text{if } (x,y) = (1,2); \\ 0.1 & \text{if } (x,y) = (3,2); \\ 0.3 & \text{if } (x,y) = (3,6); \\ 0.1 & \text{if } (x,y) = (3,7); \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that for discrete random variables X and Y,

$$E(g(X,Y)) = \sum_{(x,y):P((X,Y)=(x,y))>0} g(x,y)P((X,Y) = (x,y)) \qquad (1)$$

if g is nonnegative.

(a) (4 pts) Find $E(XY)$ using (1).

(b) (4 pts) Find $E(XY)$ using the PMF of XY.

26. (4 pts) Suppose that X has PDF f_X, and g is a function defined by

$$g(x) = \sum_{i=1}^{m} a_i I_{A_i}(x),$$

where $a_1, \ldots, a_m$ are constants and $A_1, \ldots, A_m$ are disjoint intervals. Therefore, $g(X)$ has only m possible values $a_1, \ldots, a_m$. Show that

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

27. (4 pts) Suppose that X is a random variable with MGF M_X, where

$$M_X(t) = 0.6 + 0.4e^{2t}$$

for $t \in (-\infty, \infty)$. Find $E(X^k)$ for $k \in \{1, 2, 3, 4\}$.

28. (4 pts) Suppose that Y is a discrete random variable with PMF p_Y, where

$$p_Y(y) = \begin{cases} 0.6 & \text{if } y = 0; \\ 0.4 & \text{if } y = 2; \\ 0 & \text{otherwise.} \end{cases}$$

Show that the CDF of Y is the same as the CDF of the X in Problem 27 by verifying that X and Y have the same MGF.

29. (8 pts) Suppose that X is a discrete random variable with PMF p_X, where

$$p_X(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!} I_{\{0,1,2,\ldots\}}(x) \qquad (2)$$

for $x \in R$, where $\lambda > 0$ is a constant.

(a) (4 pts) Find the MGF of X.

(b) (4 pts) Show that $E(X) = \lambda$ and find $Var(X)$.

Note. The distribution of X with the PMF p_X given in (2) is called the Poisson distribution with mean λ.

30. (8 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$$

for $x \in R$, and $\mu \in R$ and $\sigma > 0$ are constants. Let $Y = (X - \mu)/\sigma$.

(a) (4 pts) Find the MGF of Y.

(b) (4 pts) Find $E(Y^6)$.

Note. Recall that we found the MGF of X in class when $\mu = 0$ and $\sigma = 1$. You may use the MGF found in class for this problem.

31. (4 pts) Suppose that (X, Y) is a random vector with CDF $F_{X,Y}$, where for $(x, y) \in R^2$,

$$F_{X,Y}(x, y) = \begin{cases} (0.5G(x)G(y) + 0.5(1 - e^{-x})(1 - e^{-y})) & \text{if } x \geq 0 \text{ and } y \geq 0; \\ 0 & \text{otherwise,} \end{cases}$$

and the function G is defined by

$$G(x) = xI_{(0,1)}(x) + I_{[1,\infty)}(x)$$

for $x \in R$. Find $P(0 < X \leq 1 \text{ and } 1 < Y \leq 2)$.

32. (4 pts) Suppose that (X, Y, Z) is a random vector with joint CDF F. Show that

$$\begin{aligned} &P((X, Y, Z) \in (a, b] \times (c, d] \times (e, f]) \\ &= F(b, d, f) - F(b, c, f) - F(a, d, f) + F(a, c, f) \\ &-F(b, d, e) + F(b, c, e) + F(a, d, e) - F(a, c, e), \end{aligned}$$

where a, b, c, d, e, f are constants such that $a < b$, $c < d$ and $e < f$.

33. (2 pts) Consider the (X, Y) in Problem 31. Find the CDF of X.

34. Suppose that (X, Y) has PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = cxI_{(0,1)}(x)I_{(0,1)}(y)$$

for $(x, y) \in R^2$ and $c > 0$ is a constant.

(a) (2 pts) Show that $c = 2$.

(b) (4 pts) Find $P(X + 2Y \leq 1)$.

(c) (4 pts) Find a PDF of Y.

Remark. The distribution of Y is called the uniform distirbution on $(0, 1)$, denoted by $U(0, 1)$. For $a < b$, the distribution of $a + (b - a)Y$ is called the uniform distirbution on (a, b), denoted by $U(a, b)$.

35. (10 pts) Suppose that (X, Y) has PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = ce^{-(x^2+y^2)/2}I_{(0,\infty)}(x)I_{(0,\infty)}(y)$$

for $(x, y) \in R^2$ and $c > 0$ is a constant. Let $U = \sqrt{X^2 + Y^2}$ and $V = \tan^{-1}(Y/X)$. Note that for $z \in -\infty, \infty)$, $\tan^{-1}(z)$ is the value $\theta \in (-\pi/2, \pi/2)$ such that $\tan(\theta) = z$.

(a) (4 pts) Find a PDF of (U, V). Leave the constant c in your answer.

(b) (4 pts) Find a PDF of V. Leave the constant c in your answer.

(c) (2 pts) Find c using your answer in Part (b) and the fact that the integral of a PDF of V over $(-\infty, \infty)$ is 1.

Remark. The result from Problem 35(c) can be used for finding $\int_{-\infty}^{\infty} e^{-x^2/2}dx$. To see this, let $I = \int_0^{\infty} e^{-x^2/2}dx$, then

$$1 = \int_{R^2} f_{X,Y}(x, y)d(x, y) = cI^2,$$

so $I = 1/\sqrt{c}$ and $\int_{-\infty}^{\infty} e^{-x^2/2}dx = 2I = 2/\sqrt{c}$ (you should be able to obtain $2/\sqrt{c} = \sqrt{2\pi}$ if your answer for c is correct).

36. (10 pts) Let Γ be the function on $(0, \infty)$ defined by

$$\Gamma(a) = \int_0^{\infty} x^{a-1}e^{-x}dx$$

for $a > 0$. Suppose that α and β are two positive constants and (X, Y) has joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}e^{-x}I_{(0,\infty)}(x)y^{\beta-1}e^{-y}I_{(0,\infty)}(y)$$

for $(x, y) \in R^2$.

(a) (4 pts) Let M be the function on $(-\infty, 1) \times (-\infty, 1)$ defined by

$$M(t_1, t_2) = \left(\frac{1}{1 - t_1}\right)^{\alpha} \left(\frac{1}{1 - t_2}\right)^{\beta}$$

for $(t_1, t_2) \in (-\infty, 1) \times (-\infty, 1)$. Show that M is the joint MGF of (X, Y).

(b) (2 pts) Find the MGF of X.

(c) (4 pts) Find $E(XY)$ and $E(X)$.

37. (2 pts) Consider the (X, Y) in Problem 36. The distribution of X is called the gamma distribution with shape parameter α and scale parameter 1, denoted by $\Gamma(\alpha, 1)$. Show that the distribution of $(X+Y)$ is $\Gamma(\alpha+\beta, 1)$. Hint: the MGF of $(X+Y)$ can be easily obtained from the MGF of (X, Y).

38. (8 pts) Suppose that (X, Y) has a joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = cI_S(x, y)$$

for $(x, y) \in R^2$,

$$S = \{(x, y) : -2 < x + 2y < 2 \text{ and } -2 < x - 2y < 2\},$$

and $c = 1/\int_{R^2} I_S(x, y)d(x, y)$.

(a) (4 pts) Find $E(X|Y)$. You may leave c in your answer.

(b) (4 pts) Find $Var(X|Y)$. You may leave c in your answer.

39. (16 pts) Suppose that (X, Y) is a discrete random vector with PMF $p_{X,Y}$, where

$$P((X, Y) = (x, y)) = \begin{cases} 0.5 & \text{if } (x, y) = (1, 2); \\ 0.4 & \text{if } (x, y) = (0, -3); \\ 0.1 & \text{if } (x, y) = (1, -3); \\ 0 & \text{otherwise.} \end{cases}$$

(a) (8 pts) Find $E(X|Y = y)$ and $Var(X|Y = y)$ for $y \in \{2, -3\}$.

(b) (8 pts) Find $Var(E(X|Y))$, $E(Var(X|Y))$ and $Var(X)$. Verify the equality

$$Var(X) = Var(E(X|Y)) + E(Var(X|Y))$$

based on your answers.

40. (4 pts) Verify Equation (1) in the handout "Independence of two random vectors". That is, for $\mu \in R$, $\sigma > 0$, $s \in R$, $t = (t_1, t_2, t_3)^T \in R^3$, show that

$$\sum_{i=1}^{3} \left[\mu \left(t_i + \frac{s - 3\bar{t}}{3} \right) + 0.5\sigma^2 \left(t_i + \frac{s - 3\bar{t}}{3} \right)^2 \right]$$
$$= \mu \cdot s + \frac{\sigma^2 s^2}{6} + 0.5\sigma^2 \sum_{i=1}^{3} (t_i - \bar{t})^2,$$

where $\bar{t} = (t_1 + t_2 + t_3)/3$.

41. (4 pts) Suppose that (X, Y) has a joint PDF and X and Y are independent. Suppose that u and v are functions such that $E(u(X))$ and $E(v(Y))$ are finite. Show that

$$E(u(X)v(Y)) = E(u(X))E(v(Y)).$$

42. (4 pts) Consider the (X, Y) in Problem 36. Determine whehter $X + Y$ and $X - Y$ are indepedent based on the MGF of (X, Y).

43. (4 pts) Consider the (X, Y) in Problem 39. Determine whehter X and Y are independent.

44. (4 pts) Suppose that μ and σ are constants, $\sigma > 0$, and X is a random variable such that

$$E(e^{tX}) = e^{\mu t + 0.5\sigma^2 t^2}$$

for $t \in R$. Find $E(X)$ and $Var(X)$.

45. (4 pts) Suppose that X and Y are discrete random variables, X has m possible values x_1, ..., x_m, and Y has n possible values y_1, ..., y_n. Suppose that for $x \in \{x_1, \ldots, x_m\}$,

$$P(X = x|Y = y_1) = P(X = x|Y = y_2) = \cdots = P(X = x|Y = y_n).$$

Show that X and Y are independent.

Note.

- For a random variable X, if the distribution of X is $\mathcal{D}$, we will write $X \sim \mathcal{D}$.
- The normal distribution with mean μ and variance σ^2 will be denoted by $N(\mu, \sigma^2)$.

46. (4 pts) Suppose that X is a random variable and $X \sim N(\mu, \sigma^2)$. Suppose that $Y = a + bX$ for some constants a and b and $b \neq 0$. Show that $Y \sim N(E(Y), Var(Y))$. You may use the MGF of a normal distribution given in Example 4 in the handout "Independent random variables".

47. (10 pts) For $\mu \in R$ and $\sigma > 0$, let $f_{\mu,\sigma}$ be the function defined by

$$f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} \tag{3}$$

for $x \in R$. Suppose that X and Y are two random variables such that $Y \sim N(0, 1)$ and for $y \in R$, $f_{1+2y,1}$ is a version of the conditional PDF of X given $Y = y$.

(a) (6 pts) Find $E(Y|X)$.

(b) (4 pts) Determine whether X and Y are independent. Justify your answer.

Note. We will refer the $f_{\mu,\sigma}$ in (3) as the continuous PDF of $N(\mu, \sigma^2)$.

48. (8 pts) Suppose that Z_1, Z_2, Z_3 are independent random variables and $Z_i \sim N(0, 1)$ for $i = 1, 2, 3$. Let

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}.$$

(a) (4 pts) Find a PDF of (Y_1, Y_2, Y_3).

(b) (4 pts) Determine whether Y_1 and (Y_2, Y_3) are independent. Justify your answer.

49. (4 pts) Suppose that $Z_1, \ldots, Z_n$ are independent random variables and $Z_i \sim N(0,1)$ for $i = 1, 2, \ldots, n$. Suppose that A is an $n \times n$ invertible matrix of constants in R and $\mu_1, \ldots, \mu_n$ are constants in R. Let

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = A \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}.$$

Find a PDF of $(Y_1, \ldots, Y_n)$ that is determined by AA^T and $\mu_1, \ldots, \mu_n$. You may use results from linear algebra such as $\det(B^T) = \det(B)$ and

$$\det(BC) = \det(B) \cdot \det(C)$$

for square matrices B and C. Here $\det(A)$ denotes the determinant of a square matrix A.

Note. The distribution of $(Y_1, \ldots, Y_n)$ is known as a multivariate normal distribution.

50. (8 pts) Suppose that (X, Y, Z) is a random vector with joint PDF $f_{X,Y,Z}$, where

$$f_{X,Y,Z}(x, y, z) = ce^{-(x^2+4xy+5y^2)} ze^{-z} I_{(0,\infty)}(z)$$

for $(x, y, z) \in R^3$, and $c > 0$ is a constant.

(a) (6 pts) Find $E(X|Y)$ and $Var(X|Y)$.

(b) (2 pts) Find a version of the conditional PDF of Z given $(X, Y) = (x, y)$ for $(x, y) \in R^2$.

51. (6 pts) Suppose that X and Y are random variables such that $Var(X)$ and $Var(Y)$ are both finite and $Var(X) > 0$. Define a function S by

$$S(a, b) = E(Y - (a + bX))^2$$

for $(a, b) \in R^2$. Show that S is minimized when $(a, b) = (a_0, b_0)$, where $b_0 = Cov(X, Y)/Var(X)$ and $a_0 = E(Y) - b_0 E(X)$. Also, show that

$$S(a_0, b_0) = \frac{Var(X)Var(Y) - (Cov(X, Y))^2}{Var(X)}. \tag{4}$$

Note. (4) implies that $|Corr(X, Y)| \leq 1$ since $S(a_0, b_0) = Var(Y)(1 - (Corr(X, Y))^2)$.

52. (6 pts) Suppose that (X, Y) is a vector of two discrete random variables with joint PMF $p_{X,Y}$, where

$$p_{X,Y}(x, y) = \begin{cases} 0.5 & \text{if } (x, y) = (1, 2); \\ 0.4 & \text{if } (x, y) = (0, a); \\ 0.1 & \text{if } (x, y) = (-1. - 2); \\ 0 & \text{otherwise,} \end{cases}$$

and a is a constant.

(a) (4 pts) Express $Corr(X, Y)$ as a a function of a.

(b) (2 pts) Find all a's such that $|Corr(X, Y)| = 1$.

53. (6 pts) Suppose that (X, Y) is a random vector with covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 1 & a \\ a & 0.25 \end{pmatrix}$$

and a is a constant.

(a) (2 pts) Express $Corr(X, Y)$ as a function of a.

(b) (4 pts) Find a constant b such that $Var(X - bY) = 0$ when $a = 0.5$.

Note. In the above problem, it is clear that $Corr(X, Y) = 1$ when $a = 0.5$, so there is a linear relation between X and Y. Suppose that the linear relation is $X = bY + c$, then $Var(X - bY) = 0$.

54. (3 pts) Suppose that $\boldsymbol{X} = (X_1, X_2, X_3, X_4)^T$ is a random vector with covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Find $Var(X_1+X_2+X_3)$ by finding a matrix A such that $A\boldsymbol{X} = (X_1+X_2+X_3)$ and then applying Fact 2 in the handout "Covariance and correlation".

55. (4 pts) Suppose that (X, Y) has a joint PDF, f_Y is a PDF of Y and $f_{X|Y=y}$ is a version of the conditional PDF of X given $Y = y$ for $y \in S_Y = \{y : f_Y(y) > 0\}$. Then for g such that $E(g(X, Y))$ is finite, $E(g(X, Y)|Y)$ can be obtained using

$$E(g(X, Y)|Y = y) = \int g(x, y) f_{X|Y=y}(x) dx \tag{5}$$

for all $y \in S_Y$. Use (5) to show that $E(XY|Y) = YE(X|Y)$ when $E(XY)$ and $E(X)$ are finite.

56. (12 pts) Consider the random vector (X, Y, Z) in Problem 50.

(a) (2 pts) Find $E(Z|X, Y)$.

(b) (4 pts) Find a version of the conditional PDF of X given $Y = y$ for $y \in R$.

(c) (4 pts) Find $Cov(X, Y)$.

(d) (2 pts) Determine whether X and Y independent and justify your answer.

57. (4 pts) Suppose that X and Y are random variables such that $Y \sim N(0, 1)$ and a version of the conditional PDF of X given $Y = y$ is g_y, where g_y is the function $f_{\mu,\sigma}$ given in (3) with $\mu = y$ and $\sigma = 1$. Find a version of the conditional PDF of Y given $X = x$ for $x \in R$.

58. (6 pts) Suppose that $\boldsymbol{X} = (X_1, \ldots, X_m)^T$ and $Y = (Y_1, \ldots, Y_n)^T$ are two random vectors. Then the covariance matrix for the pair $(\boldsymbol{X}, \boldsymbol{Y})$ is defined to be the matrix

$$E((\boldsymbol{X} - \boldsymbol{\mu}_1)(\boldsymbol{Y} - \boldsymbol{\mu}_2)^T),$$

where $\boldsymbol{\mu}_1 = E(\boldsymbol{X})$ and $\boldsymbol{\mu}_2 = E(\boldsymbol{Y})$. We denote the covariance matrix for $(\boldsymbol{X}, \boldsymbol{Y})$ by $Cov(\boldsymbol{X}, \boldsymbol{Y})$.

(a) (3 pts) Suppose that $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant column vectors of length m and n respectively. Show that

$$Cov(\boldsymbol{X}+\boldsymbol{a},\boldsymbol{Y}) = Cov(\boldsymbol{X},\boldsymbol{Y}) = Cov(\boldsymbol{X},\boldsymbol{Y}+\boldsymbol{b}).$$

(b) (3 pts) Suppose that A and B are constant matrice with numbers of columns m and n respectively. Show that

$$Cov(A\boldsymbol{X},B\boldsymbol{Y}) = ACov(\boldsymbol{X},\boldsymbol{Y})B^T.$$

Note.

- The (i,j)-th element of $Cov(\boldsymbol{X},\boldsymbol{Y})$ is $Cov(X_i,Y_j)$ for $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$.
- $Cov(\boldsymbol{X},\boldsymbol{X})$ is the covariance matrix of $\boldsymbol{X}$ defined in the handout "Covariance and correlation".
- Based on the results of this problem, the covariance matrix of $A\boldsymbol{X}+\boldsymbol{b}$ is

$$Cov(A\boldsymbol{X}+\boldsymbol{b},A\boldsymbol{X}+\boldsymbol{b}) = ACov(\boldsymbol{X},\boldsymbol{X})A^T,$$

which is the result given in Fact 2 in the handout "Covariance and correlation".

59. (6 pts) Suppose that $\boldsymbol{X} = (X_1,X_2,X_3,X_4)^T$ is a random vector with $E(\boldsymbol{X}) = (0,0,0,0)^T$ and covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(a) (3 pts) Find the best linear predictor of $(X_1,X_4)^T$ based on X_2 and X_3.

(b) (3 pts) Suppose that $\boldsymbol{X} \sim N(E(\boldsymbol{X}),\Sigma)$. Find $E(X_1X_2|X_3,X_4)$.

60. (3 pts) Suppose that Y, X_1, …, X_k are random variables with finite means and variances. For $(b_1,\ldots,b_k) \in R^k$, define

$$S(b_1,\ldots,b_k) = E[(Y-(b_1X_1+\cdots+b_kX_k))^2].$$

Show that

$$\frac{\partial}{\partial b_i}S(b_1,\ldots,b_k) = E\left[\frac{\partial}{\partial b_i}(Y-(b_1X_1+\cdots+b_kX_k))^2\right]$$

for $i \in \{1,\ldots,k\}$.

Hint: express $(Y-(b_1X_1+\cdots+b_kX_k))^2$ as $(b_iX_i+U_i)^2$ for some random variable U_i that does not depend on b_i.

61. (6 pts) Suppose that $\boldsymbol{Y}$, $\boldsymbol{X}$ and $\boldsymbol{U}$ are random vectors in R^m, R^n, R^m respectively such that $\boldsymbol{X}$ and $\boldsymbol{U}$ are independent and

$$\boldsymbol{Y} = g(\boldsymbol{X}) + \boldsymbol{U} \qquad (6)$$

for some function $g : R^n \to R^m$. Suppse that g is differentiable, and $\boldsymbol{X}$ has a PDF $f_{\boldsymbol{X}}$ that is positive on R^n. Suppose that $f_{\boldsymbol{U}}$ is a PDF $\boldsymbol{U}$ respectively. For $\boldsymbol{x} \in R^n$, define

$$f_{\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}}(\boldsymbol{y}) = f_{\boldsymbol{U}}(\boldsymbol{y} - g(\boldsymbol{x})) \tag{7}$$

for $\boldsymbol{y} \in R^m$. Show that $f_{\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}}$ is a version of the conditional PDF of $\boldsymbol{Y}$ given $\boldsymbol{X} = \boldsymbol{x}$ for $\boldsymbol{x} \in R^n$.

Note. When the distribution of $(\boldsymbol{X}, \boldsymbol{Y})$ is a multivariate normal distribution, let $g(\boldsymbol{X})$ be the best linear predictor of $\boldsymbol{Y}$ based on $\boldsymbol{X}$, then (6) holds with $\boldsymbol{U} = \boldsymbol{Y} - g(\boldsymbol{X})$ and $\boldsymbol{X}$ and $\boldsymbol{U}$ are independent by Fact 3 in the handout "Multivariate normal distributions". From the result of Problem 61, a version of the conditional PDF of $\boldsymbol{Y}$ given $\boldsymbol{X} = \boldsymbol{x}$ is the $f_{\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}}$ given in (7) with $f_{\boldsymbol{U}}$ being the $N(\boldsymbol{\mu}, \Sigma)$ PDF in Equation (2) in the handout "Multivariate normal distributions", where $\boldsymbol{\mu} = (0, \ldots, 0)^T \in R^m$ and Σ is the covariance matrix of $\boldsymbol{U}$. This the result stated in Fact 4 (ii) in the handout "Multivariate normal distributions".

62. (8 pts) Suppose that $\boldsymbol{X} = (X_1, X_2, X_3, X_4, X_5)^T$ is a random vector with $E(\boldsymbol{X}) = (1, 2, 3, 4, 5)^T$ and covariance matrix Σ_0, where

$$\Sigma_0 = \begin{pmatrix} 2 & 1 & 0.5 & 0 & 0 \\ 1 & 2 & 1 & 0.5 & 0 \\ 0.5 & 1 & 2 & 0 & 0 \\ 0 & 0.5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Suppose that $\boldsymbol{X} \sim N(E(\boldsymbol{X}), \Sigma_0)$. For $\boldsymbol{\mu} \in R^m$ and an $m \times m$ covariance matrix Σ, let $f_{\boldsymbol{\mu},\Sigma}$ be the PDF of $N(\boldsymbol{\mu}, \Sigma)$ defined by

$$f_{\boldsymbol{\mu},\Sigma}(\boldsymbol{x}) = (2\pi)^{-m/2} (\det(\Sigma))^{-1/2} e^{-(\boldsymbol{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu})/2} \tag{8}$$

for $\boldsymbol{x} \in R^m$.

(a) (6 pts) For $(x_3, x_4, x_5) \in R^3$, find $\boldsymbol{\mu}$ and Σ so that the PDF $f_{\boldsymbol{\mu},\Sigma}$ given in (8) is a version of the conditional PDF of (X_1, X_2) given $(X_3, X_4, X_5) = (x_3, x_4, x_5)$.

(b) (2 pts) Find a version of the conditional PDF of X_5 given $(X_1, X_2, X_3, X_4) = (x_1, x_2, x_3, x_4)$ for $(x_1, x_2, x_3, x_4) \in R^4$.

63. (4 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$, where $\mu \in R$ and $\sigma > 0$ are unknown. Let $\bar{X} = \sum_{i=1}^n X_i/n$, $\bar{Y} = \sum_{i=1}^n X_i^2/n$, and $\bar{Z} = \sum_{i=1}^n (X_i - \mu)^2/n$. Which of the following statements are true? You may write down your answers directly without justification.

(a) $\bar{X}$ is a consistent estimator of μ.

(b) $\bar{Y} - (\bar{X})^2$ is a consistent estimator of σ^2.

(c) $\bar{Z}$ is a consistent estimator of σ^2.

(d) $\bar{Z}$ converges to σ^2 in probability as $n \to \infty$.

64. (4 pts) Suppose that $\{X_n\}_{n=1}^\infty$ and $\{Y_n\}_{n=1}^\infty$ are two sequences of random variables such that $X_n \xrightarrow{P} \mu_1$ and $Y_n \xrightarrow{P} \mu_2$ as $n \to \infty$, where μ_1 and μ_2 are constants. Show that

$$\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \xrightarrow{P} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \text{ as } n \to \infty.$$

Hint: find a constant $k > 0$ such that

$$\sqrt{(X_n - \mu_1)^2 + (Y_n - \mu_2)^2} \leq k\,(|X_n - \mu_1| + |Y_n - \mu_2|)$$

and then use the inequality to provide an upper bound for

$$P(\sqrt{(X_n - \mu_1)^2 + (Y_n - \mu_2)^2} > \varepsilon)$$

using $P(|X_n - \mu_1| > \varepsilon_1)$ and $P(|Y_n - \mu_2| > \varepsilon_2)$ for some properly chosen ε_1 and ε_2.

65. (4 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $U(0, \theta)$, where $\theta > 0$. Find a consistent estimator of $1/\theta$ and justify your answer.