Homework Problems

- Note. Always show your work in your homework solutions to receive full points unless it is stated otherwise.
- Notation. We will use the notation I_A to denote the indicator function of A for a given set A, which is defined by

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{otherwise.} \end{cases}$$

1. (4 pts) Suppose that $\mathcal{F}$ is a σ-field on a space Ω and $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets in $\mathcal{F}$. Show that

$$(\cup_{n=1}^{\infty} A_n)^c = \cap_{n=1}^{\infty}(A_n^c).$$

2. (8 pts) Suppose that $\Omega = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. Let $\mathcal{C} = \{\emptyset, \Omega, A, B\}$. Find $\sigma(\mathcal{C})$, the σ-field on Ω that is generated by $\mathcal{C}$. List all sets that should be included in $\sigma(\mathcal{C})$ and explain why they should be in $\sigma(\mathcal{C})$. You do not have to verify that the collection of sets in your list is a σ-field, but be sure that it is.

3. (8 pts) Suppose that X is a discrete random variable with PMF p_X, which is given below:

$$p_X(x) = \begin{cases} 0.2 & \text{if } x = -1; \\ 0.4 & \text{if } x = 0; \\ c \cdot (0.5)^x & \text{if } x \in \{1, 2, 3 \ldots, \}; \\ 0 & \text{otherwise,} \end{cases}$$

where $c > 0$ is a constant.

   (a) (4 pts) Find c.
   (b) (4 pts) Find $P(X > 25)$.

4. (8 pts) Suppose that X and Y are discrete random variables, and

$$P((X,Y) = (x,y)) = \begin{cases} 0.5 & \text{if } (x,y) = (1,2); \\ 0.1 & \text{if } (x,y) = (3,2); \\ 0.3 & \text{if } (x,y) = (3,6); \\ 0.1 & \text{if } (x,y) = (3,7); \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that for discrete random variables X and Y,

$$E(g(X,Y)) = \sum_{(x,y):P((X,Y)=(x,y))>0} g(x,y)P((X,Y) = (x,y)) \tag{1}$$

if g is nonnegative.

   (a) (4 pts) Find $E(XY)$ using (1).
   (b) (4 pts) Find $E(XY)$ using the PMF of XY.

5. (8 pts) Suppose that X is a discrete random variable with PMF p_X, where

$$p_X(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!} I_{\{0,1,2,\ldots\}}(x) \tag{2}$$

for $x \in R$, where $\lambda > 0$ is a constant.

(a) (4 pts) Show that $E(X) = \lambda$ and $E(X(X-1)) = \lambda^2$.

(b) (4 pts) Find $Var(X)$. You may use the result in Part (a) even if you choose not to do Part (a).

Note. The distribution of X with the PMF p_X given in (2) is called the Poisson distribution with mean λ.

6. (8 pts) For $a, b \in R$ such that $a < b$, define a function $f_{a,b}$ on R as follows: for $x \in R$,

$$f_{a,b}(x) = I_{(a,b)}(x) = \begin{cases} 1/(b-a) & \text{if } x \in (a,b); \\ 0 & \text{if } x \notin (a,b). \end{cases}$$

Suppose that X is a random variable with PDF $f_{0,1}$.

(a) (4 pts) Find $P(X > 0.6)$.

(b) (4 pts) Find $E(X^2)$.

Note.

- For a random variable X with PDF $f_{a,b}$ defined in Problem 6, the distribution of X is called the uniform distribution on (a,b), denoted by $U(a,b)$.
- We will write $X \sim U(a,b)$ to indicate the distribution of X is $U(a,b)$.

7. (8 pts) Suppose that $\mathcal{F}$ is a σ-field on $\Omega = \{1,2,3,4,5\}$. Let $A = \{1,2,4,5\}$, $B = \{1,2,4\}$ and $C = \{1,4,5\}$. Suppose that A, B, C are in $\mathcal{F}$ and P is a probability function defined on $\mathcal{F}$.

(a) (4 pts) Find disjoint sets D_1, D_2, D_3 in $\mathcal{F}$ such that each of A, B and C can be expressed as the union of some sets in $\{D_1, D_2, D_3\}$. Write down those expressions.

(b) (4 pts) Explain why we cannot have $(P(A), P(B), P(C)) = (0.5, 0.3, 0.1)$.

8. (6 pts) Suppose that $\mathcal{F}$ is a σ-field on a space Ω and P is a probability function on $\mathcal{F}$. Suppose that $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets in $\mathcal{F}$, i.e. $A_n \supset A_{n+1}$ for all $n \in \{1,2,\ldots\}$. Show that

$$P\left(\cap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

You may use the following fact in your proof.

Fact 1 Suppose that $\mathcal{F}$ is a σ-field on a space Ω and P is a probability function on $\mathcal{F}$. Suppose that $\{A_n\}_{n=1}^{\infty}$ is an increasing sequence of sets in $\mathcal{F}$, i.e. $A_n \subset A_{n+1}$ for all $n \in \{1,2,\ldots\}$. Then,

$$P\left(\cup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

9. (2 pts) Suppose that P is a probability function on $\mathcal{B}(R)$ such that for $n \in \{1,2,\ldots,\}$,

$$P\left(\left(-\frac{1}{n}, \frac{1}{n}\right)\right) = 0.4 + \frac{0.6}{n}.$$

Find $P(\{0\})$.

10. (6 pts) Suppose that F is the CDF of a random variable X, and $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence of real numbers such that $\lim_{n\to\infty} a_n = \infty$. Show that

$$\lim_{n\to\infty} F(-a_n) = 0$$

without using the fact that $\lim_{x\to-\infty} F(x) = 0$.

Remark. It can be shown that $\lim_{x\to-\infty} F(x)$ exists, so the result in Problem 10 implies that $\lim_{x\to-\infty} F(x)$ is 0.

11. (6 pts) Suppose that $\mathcal{F}$ is a σ-field on a space Ω and P is a probability function defined on $\mathcal{F}$. For a set $B \in \mathcal{F}$ such that $P(B) > 0$, define a function Q on $\mathcal{F}$ by

$$Q(A) = \frac{P(A \cap B)}{P(B)}$$

for $A \in \mathcal{F}$. Verify that Q is a probabilty function on $\mathcal{F}$.

12. (14 pts) Suppose that X is a random variable with CDF F, where

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ 0.5 + 0.5x & \text{if } 0 < x < 1; \\ 1 & \text{if } x > 1; \end{cases}$$

    (a) (2 pts) Find $F(0)$.
    (b) (2 pts) Find $F(1)$.
    (c) (4 pts) Find $P(X = a)$ for every $a \in R$.
    (d) (3 pts) Find $P(0 \le X \le 1)$.
    (e) (3 pts) Determine whether X is a discrete random variable. If so, find all the possible value(s) of X. If not, explain why.

13. (6 pts) Suppose that $X \sim U(a, b)$, where $a < b$. Show that $(X - a)/(b - a) \sim U(0, 1)$. A PDF of $U(a, b)$ is the $f_{a,b}$ given in Problem 6.

14. (6 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = 2xe^{-x^2} I_{(0,\infty)}(x) \text{ for } x \in R.$$

Find a PDF of $Y = \sqrt{X}$.

15. (6 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = |x| I_{(-1,0)}(x) + 0.5e^{-x} I_{(0,\infty)}(x)$$

Find a PDF of $Y = X^2$.

16. (6 pts) Suppose that X is a random variable with CDF F, where

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ 1 - e^{-2x} & \text{if } x \ge 0. \end{cases}$$

    (a) (2 pts) Find $P(X > 2)$.
    (b) (4 pts) Find a PDF of X.

17. (8 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = 2xe^{-x^2} I_{(0,\infty)}(x) \text{ for } x \in R.$$

(a) (4 pts) Find the CDF of X.

(b) (4 pts) Find the median and the IQR of the distribution of X.

18. (8 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = e^{-x} I_{(0,\infty)}(x) \text{ for } x \in R.$$

Suppose that $Y = g(X)$, where

$$g(x) = x \cdot I_{(-\infty,0.5]}(x) + 0.5 \cdot I_{(0.5,\infty)}(x)$$

for $x \in R$.

(a) (4 pts) Find the CDF of Y.

(b) (2 pts) Explain why Y does not have a PDF.

(c) (2 pts) Find the quantile of order $(1 - e^{-0.5})$ of the distribution of Y.

19. (8 pts) Suppose that X is a random variable such that both $E(X^2)$ and $E(|X|)$ are finite. Recall that $Var(X) = E(Y)$ with $Y = (X - E(X))^2$. Use Properties (i)–(iii) listed in Page 6 of the handout "Quantile and expectation" to prove the following results:

(a) (4 pts) $Var(X) = E(X^2) - (E(X))^2$;

(b) (4 pts) $Var(cX) = c^2 Var(X)$ for a constant c.

20. (4 pts) Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = \frac{1}{2x^2} \cdot I_{(-\infty,-1)\cup(1,\infty)}(x)$$

for $x \in R$.

(a) (2 pts) Determine whether $X \cdot I_{(0,\infty)}(X)$ is integrable and justify your answer.

(b) (2 pts) Determine whether $E(X)$ can be defined and justify your answer.

21. (4 pts) Suppose that X and Y are random variables and the random vector (X, Y) has PDF $f_{X,Y}$. Let

$$S = \{(x, y) \in R^2 : f_{X,Y}(x, y) > 0\}.$$

Suppose that S is an open set, (x_0, y_0) is a point in S such that $f_{X,Y}$ is continuous and positive on

$$B((x_0, y_0), \delta) \stackrel{\text{def}}{=} \{(x, y) \in R^2 : \|(x, y) - (x_0, y_0)\| < \delta\}$$

for some $\delta > 0$, where $\|\cdot\|$ is the Euclidean norm. Suppose that u and v are two real-valued differentiable functions on S. Let

$$g(x, y) = (u(x, y), v(x, y))$$

for $(x, y) \in S$. Suppose that g is one-to-one on S. Let $U = u(X, Y)$ and $V = v(X, Y)$. Suppose that (U, V) has a PDF $f_{U,V}$ that is continuous and positive on

$$B(g(x_0, y_0), \varepsilon) \stackrel{\text{def}}{=} \{(u, v) \in R^2 : \|(u, v) - g(x_0, y_0)\| < \varepsilon\}$$

for some $\varepsilon > 0$. Let $(u_0, v_0) = g(x_0, y_0)$. Let

$$J(x,y) = \begin{pmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{pmatrix}$$

for $(x,y) \in S$. Suppose that the partial derivative functions u_x, u_y, v_x and v_y are continuous on $B((x_0,y_0),\delta)$ and the determinant of $J(x,y) \neq 0$ for $(x,y) \in B((x_0,y_0),\delta)$. For $h > 0$, let R_h be the region inside the parallelogram $ABCD$, where $A = (u_0, v_0)$, $B = (u_0 + u_x(x_0,y_0)h, v_0 + v_x(x_0,y_0)h)$,

$$C = (u_0 + u_x(x_0,y_0)h + u_y(x_0,y_0)h, v_0 + v_x(x_0,y_0)h + v_y(x_0,y_0)h)$$

and $D = (u_0 + u_y(x_0,y_0)h, v_0 + v_y(x_0,y_0)h)$. Then it is clear that for small enough $h > 0$,

$$(x_0, x_0 + h) \times (y_0, y_0 + h) \subset B((x_0,y_0),\delta)$$

and

$$R_h \subset B(g(x_0,y_0),\varepsilon).$$

Use the result

$$\lim_{h \to 0^+} \frac{P((X,Y) \in (x_0, x_0+h) \times (y_0, y_0+h))}{h^2} = \lim_{h \to 0^+} \frac{P((U,V) \in R_h)}{h^2}$$

to deduce that

$$f_{X,Y}(x_0,y_0) = f_{U,V}(u_0,v_0) \cdot |\text{ determinant of } J(x_0,y_0)| \, .$$

22. (4 pts) Suppose that X is a random variable with finite expectation μ and standard deviation $\sigma > 0$. Let $Y = (X - \mu)/\sigma$. Find $E(Y)$ and $Var(Y)$ with $\mu = 1.5$ and $\sigma = 1.2$.

23. (4 pts) Suppose that X is a random variable with MGF M_X, where

$$M_X(t) = 0.6 + 0.4e^{2t}$$

for $t \in (-\infty, \infty)$. Find $E(X^k)$ for $k \in \{1,2,3,4\}$.

Note. You may use the following result.

Fact 2 Suppose that

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

for $|x| < r$ for some positive constant r, then

$$f^{(k)}(0) = a_k \cdot k!$$

for $k \in \{0,1,2,\ldots\}$.

24. (4 pts) Suppose that Y is a discrete random variable with PMF p_Y, where

$$p_Y(y) = \begin{cases} 0.6 & \text{if } y = 0; \\ 0.4 & \text{if } y = 2; \\ 0 & \text{otherwise.} \end{cases}$$

Show that the distribution of Y is the same as the distribution of the X in Problem 23 by verifying that X and Y have the same MGF.

25. (8 pts) Suppose that X is a random variable whose distribution is the Poisson distribution with mean λ, where $\lambda > 0$. The PMF of X is the function p_X given in (2) in Problem 5.

   (a) (4 pts) Find the MGF of X.

   (b) (4 pts) Show that $E(X) = \lambda$ and find $Var(X)$ using the MGF of X.

26. (4 pts) Suppose that X is a random variable with MGF M_X, where $M_X(t) < \infty$ for $t \in (-h, h)$ for some positive constant h. For constants a and $b \in R$, let $Y = a + bX$ and let M_Y be the MGF of Y. Show that if $b \neq 0$, then

$$M_Y(t) = e^{ta} M_X(tb) < \infty$$

for $t \in (-h/|b|, h/|b|)$.

27. (16 pts) For constants $\mu \in R$ and $\sigma > 0$, define

$$f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$$

for $x \in R$. Suppose that X and Z are random variables with PDF $f_{\mu,\sigma}$ and $f_{0,1}$, respectively.

   (a) (4 pts) Let M_Z be the MGF of Z. Show that

$$M_Z(t) = e^{0.5t^2}$$

   for $t \in R$, $E(Z) = 0$ and $Var(Z) = 1$.

   (b) (4 pts) Find $E(Z^6)$.

   (c) (4 pts) Let $Y = (X - \mu)/\sigma$. Show that Y and Z have the same distribution.

   (d) (4 pts) Find the MGF of X and deduce that $E(X) = \mu$ and $Var(X) = \sigma^2$.

Note. The distribution of X in Problem 27 is called the normal distribution with mean μ and variance σ^2, denoted by $N(\mu, \sigma^2)$. When $(\mu, \sigma) = (0, 1)$, $N(0, 1)$ is called the standard normal distribution.

28. (8 pts) Suppose that (X, Y) is a random vector with CDF $F_{X,Y}$, where for $(x, y) \in R^2$,

$$F_{X,Y}(x, y) = 0.5G(x)G(y) + 0.5(1 - e^{-x})(1 - e^{-y}) I_{[0,\infty)}(x) I_{[0,\infty)}(y),$$

and the function G is defined by

$$G(x) = x I_{(0,1)}(x) + I_{[1,\infty)}(x)$$

for $x \in R$.

   (a) (4 pts) Find $P(0 < X \leq 1 \text{ and } 1 < Y \leq 2)$.

   (b) (4 pts) Find the CDF of X.

29. (4 pts) Suppose that (X, Y, Z) is a random vector with joint CDF F. Show that

$$\begin{aligned}
&P((X, Y, Z) \in (a, b] \times (c, d] \times (e, f]) \\
&= F(b, d, f) - F(b, c, f) - F(a, d, f) + F(a, c, f) \\
&-F(b, d, e) + F(b, c, e) + F(a, d, e) - F(a, c, e),
\end{aligned}$$

where a, b, c, d, e, f are constants such that $a < b$, $c < d$ and $e < f$.

30. (12 pts) Suppose that (X, Y) has PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = c \cdot x I_{(0,1)}(x) I_{(0,1)}(y)$$

for $(x, y) \in R^2$ and $c > 0$ is a constant.

(a) (4 pts) Show that $c = 2$.

(b) (4 pts) Find $P(X + 2Y \leq 1)$.

(c) (4 pts) Find a PDF of Y.

31. (4 pts) Suppose that (X, Y) has joint PDF $f_{X,Y}$ and there exist two non-negative functions g and h such that

$$f_{X,Y}(x, y) = g(x)h(y)$$

for $(x, y) \in R^2$. Show that there exist f_X: a PDF of X and f_Y: a PDF of Y such that

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \tag{3}$$

for $(x, y) \in R^2$.

Remark. (3) implies that

$$\begin{aligned} & P(\{X \in A\} \cap \{Y \in B\}) \\ &= \int_{A \times B} f_X(x) f_Y(y) d(x, y) \\ &= \int_A f_X(x) dx \cdot \int_B f_Y(y) dy \\ &= P(\{X \in A\}) P(\{Y \in B\}) \end{aligned}$$

for A, $B \in \mathcal{B}(R)$, which implies that X and Y are independent.

32. (4 pts) Suppose that X and Y are random variables on a sample space Ω. It can be shown that the limit $\lim_{y \to \infty} P(X \leq x \text{ and } Y \leq y)$ exists for every $x \in R$. Show that

$$\lim_{y \to \infty} P(X \leq x \text{ and } Y \leq y) = P(X \leq x)$$

for $x \in R$.

33. (10 pts) Suppose that (X, Y) has PDF $f_{X,Y}$, where

$$f_{X,Y}(x, y) = c e^{-(x^2+y^2)/2} I_{(0,\infty)}(x) I_{(0,\infty)}(y)$$

for $(x, y) \in R^2$ and $c > 0$ is a constant. Let $U = \sqrt{X^2 + Y^2}$ and $V = \tan^{-1}(Y/X)$. Note that for $z \in (-\infty, \infty)$, $\tan^{-1}(z)$ is the value $\theta \in (-\pi/2, \pi/2)$ such that $\tan(\theta) = z$.

(a) (4 pts) Find a PDF of (U, V). Leave the constant c in your answer.

(b) (4 pts) Find a PDF of V. Leave the constant c in your answer.

(c) (2 pts) Find c using your answer in Part (b) and the fact that the integral of a PDF of V over $(-\infty, \infty)$ is 1.

Remark. The result from Problem 33(c) can be used for finding $\int_{-\infty}^{\infty} e^{-x^2/2}dx$. To see this, let $I = \int_0^{\infty} e^{-x^2/2}dx$, then

$$1 = \int_{R^2} f_{X,Y}(x,y)d(x,y) = cI^2,$$

so $I = 1/\sqrt{c}$ and $\int_{-\infty}^{\infty} e^{-x^2/2}dx = 2I = 2/\sqrt{c}$ (you should be able to obtain $2/\sqrt{c} = \sqrt{2\pi}$ if your answer for c is correct).

34. (10 pts) Let Γ be the function on $(0,\infty)$ defined by

$$\Gamma(a) = \int_0^{\infty} x^{a-1}e^{-x}dx$$

for $a > 0$. Suppose that α and β are two positive constants and (X,Y) has joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}e^{-x}I_{(0,\infty)}(x)y^{\beta-1}e^{-y}I_{(0,\infty)}(y)$$

for $(x,y) \in R^2$.

(a) (4 pts) Let M be the function on $(-\infty,1)\times(-\infty,1)$ defined by

$$M(t_1,t_2) = \left(\frac{1}{1-t_1}\right)^{\alpha}\left(\frac{1}{1-t_2}\right)^{\beta}$$

for $(t_1,t_2) \in (-\infty,1)\times(-\infty,1)$. Show that M is the joint MGF of (X,Y).

(b) (2 pts) Find the MGF of X.

(c) (4 pts) Find $E(XY)$ and $E(X)$.

35. (2 pts) Consider the (X,Y) in Problem 34. The distribution of X is called the gamma distribution with shape parameter α and scale parameter 1, denoted by $\Gamma(\alpha,1)$. Show that the distribution of $(X+Y)$ is $\Gamma(\alpha+\beta,1)$. Hint: the MGF of $(X+Y)$ can be easily obtained from the MGF of (X,Y).

36. (8 pts) Suppose that (X,Y) has a joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x,y) = cI_S(x,y)$$

for $(x,y) \in R^2$,

$$S = \{(x,y) : -2 < x+2y < 2 \text{ and } -2 < x-2y < 2\},$$

and $c = 1/\int_{R^2} I_S(x,y)d(x,y)$.

(a) (4 pts) Find $P((X+2Y) > 0)$. You may leave c in your answer.

(b) (4 pts) Find c.

37. (8 pts) Suppose that (X,Y) has a joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x,y) = cI_S(x,y)$$

for $(x,y) \in R^2$,

$$S = \{(x,y) : -2 < x+2y < 2 \text{ and } -2 < x-2y < 2\},$$

and $c = 1/\int_{R^2} I_S(x,y)d(x,y)$.

(a) (4 pts) Find $E(X|Y)$. You may leave c in your answer.

(b) (4 pts) Find $Var(X|Y)$. You may leave c in your answer.

38. (16 pts) Suppose that (X, Y) is a discrete random vector with PMF $p_{X,Y}$, where

$$P((X,Y) = (x,y)) = \begin{cases} 0.5 & \text{if } (x,y) = (1,2); \\ 0.4 & \text{if } (x,y) = (0,-3); \\ 0.1 & \text{if } (x,y) = (1,-3); \\ 0 & \text{otherwise.} \end{cases}$$

(a) (8 pts) Find $E(X|Y = y)$ and $Var(X|Y = y)$ for $y \in \{2, -3\}$.

(b) (8 pts) Find $Var(E(X|Y))$, $E(Var(X|Y))$ and $Var(X)$. Verify the equality

$$Var(X) = Var(E(X|Y)) + E(Var(X|Y))$$

based on your answers.

39. (8 pts) Suppose that (X, Y) has joint PDF $f_{X,Y}$.

(a) (4 pts) For random variables $g(X, Y)$ and $h(Y)$, show that

$$E(g(X,Y)h(Y)|Y) = h(Y)E(g(X,Y)|Y).$$

(b) (4 pts) For random variables $g_1(X, Y)$ and $g_2(X, Y)$, show that

$$E((g_1(X,Y) + g_2(X,Y))|Y) = E(g_1(X,Y)|Y) + E(g_2(X,Y)|Y).$$

40. (16 pts) For $\mu \in R$ and $\sigma > 0$, let $f_{\mu,\sigma}$ be the PDF of $N(\mu, \sigma^2)$ given in Problem 27. Suppose that X and Y are random variables, Y has PDF $f_{0,1}$, and $\{f_{1+2y,1} : y \in R\}$ is a version of the conditional PDF of X given Y.

(a) (6 pts) Find a PDF of X.

(b) (6 pts) Find $E(Y|X)$.

(c) (4 pts) Determine whether X and Y are independent. Justify your answer.

41. (6 pts) Suppose that X and Y are independent discrete random variables. Suppose that u and v are functions such that $E(u(X))$ and $E(v(Y))$ are finite. Show that

$$E(u(X)v(Y)) = E(u(X))E(v(Y)).$$

42. (6 pts) Consider the (X, Y) in Problem 34. Determine whehter $X + Y$ and $X - Y$ are indepedent based on the MGF of (X, Y).

43. (4 pts) Consider the (X, Y) in Problem 38. Determine whehter X and Y are independent.

44. (4 pts) Suppose that X and Y are discrete random variables, X has m possible values $x_1, \ldots, x_m$, and Y has n possible values $y_1, \ldots, y_n$. Suppose that for $x \in \{x_1, \ldots, x_m\}$,

$$P(X = x|Y = y_1) = P(X = x|Y = y_2) = \cdots = P(X = x|Y = y_n).$$

Show that X and Y are independent.

45. (8 pts) Suppose that Z_1, Z_2, Z_3 are independent random variables and $Z_i \sim N(0,1)$ for $i = 1, 2, 3$. Let

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}.$$

(a) (4 pts) Find a PDF of (Y_1, Y_2, Y_3).

(b) (4 pts) Determine whether Y_1 and (Y_2, Y_3) are independent. Justify your answer.

46. (6 pts) Suppose that $X_1, \ldots, X_n$ are IID and $X_1 \sim N(\mu, \sigma^2)$. Let $\bar{X} = \sum_{i=1}^n X_i/n$ and $Y = (X_1 - \bar{X}, \ldots, X_n - \bar{X})^T$. Show that $\bar{X}$ and Y are independent.

47. (6 pts) Suppose that $Z_1, \ldots, Z_m$ are IID and $Z_1 \sim N(0,1)$. Let $U = \sum_{i=1}^m Z_i^2$. Show that $U/2 \sim \Gamma(m/2, 1)$.

Note.

- Recall that for $\alpha > 0$, the MGF of the gamma distribution $\Gamma(\alpha, 1)$ is the MGF of the random variable X in Problem 34.
- To solve this problem, you may use the results in Problem 34.
- For $\mu \in R$ and $\sigma > 0$, a PDF of $N(\mu, \sigma^2)$ is the $f_{\mu,\sigma}$ given in Problem 27.

48. (4 pts) Suppose that W is an $n \times m$ matrix of random variables, and B is an $m \times k$ non-random matrix. Show that $E(WB) = E(W)B$.

49. (4 pts) Suppose that $\boldsymbol{X} = (X_1, \ldots, X_m)^T$ is a random vector. Let Σ be the covariance matrix of $\boldsymbol{X}$. Suppose that A is a $n \times m$ non-random matrix. Show that the covariance matrix of $A\boldsymbol{X}$ is $A\Sigma A^T$. You may use the fact that for a random vector $\boldsymbol{Z}$, the covariance matrix of $\boldsymbol{Z}$ is $E(\boldsymbol{Z}_*\boldsymbol{Z}_*^T)$, where $\boldsymbol{Z}_* = \boldsymbol{Z} - E(\boldsymbol{Z})$.

50. (6 pts) Prove the following result:

Fact 3 Suppose that the distribution of $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)^T$ is multivariate normal. If $Cov(X_i, Y_j) = 0$ for $1 \le i \le m$, $1 \le j \le n$, then the two random vectors $(X_1, \ldots, X_m)^T$ and $(Y_1, \ldots, Y_n)^T$ are independent.

51. (6 pts) Suppose that

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 & 2 & 5 \\ 2 & 16 & 3 \\ 5 & 3 & 25 \end{pmatrix} \right).$$

(a) (3 pts) Find a constant b such that $Y - bX$ is independent of X.

(b) (3 pts) Find constants c and d such that $Z - cY - dX$ is independent of (X, Y).

52. (6 pts) Suppose that $Z_1, \ldots, Z_n$ are independent random variables and $Z_i \sim N(0,1)$ for $i = 1, 2, \ldots, n$. Suppose that A is an $n \times n$ invertible matrix of constants in R and $\mu_1, \ldots, \mu_n$ are constants in R. Let

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = A \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}.$$

Find a PDF of $(Y_1, \ldots, Y_n)$ that is determined by AA^T and $\mu_1, \ldots, \mu_n$. You may use results from linear algebra such as $\det(B^T) = \det(B)$ and

$$\det(BC) = \det(B) \cdot \det(C)$$

for square matrices B and C. Here $\det(A)$ denotes the determinant of a square matrix A.

Note. In Problem 52, AA^T and $(\mu_1, \ldots, \mu_n)^T$ are the covariance matrix and the mean vector of $(Y_1, \ldots, Y_n)^T$, respectively.

53. (3 pts) Suppose that U and X are random variables such that $E(X^2) < \infty$ and $E(U^2) < \infty$. Let $S(b) = E(U - bX)^2$, where b is a constant in R. Verify that

$$\frac{d}{db} S(b) = E\left[\frac{d}{db}(U - bX)^2\right].$$

Remark. The result in Problem 53 implies that

$$\frac{\partial}{\partial b_i} E[(Y - (a + b_1 X_1 + \cdots + b_k X_k))^2] = E\left[\frac{\partial}{\partial b_i}(Y - (a + b_1 X_1 + \cdots + b_k X_k))^2\right]$$

for $i \in \{1, \ldots, k\}$ and

$$\frac{\partial}{\partial a} E[(Y - (a + b_1 X_1 + \cdots + b_k X_k))^2] = E\left[\frac{\partial}{\partial a}(Y - (a + b_1 X_1 + \cdots + b_k X_k))^2\right].$$

54. (4 pts) Suppose that $\boldsymbol{X} = (X_1, X_2, X_3, X_4)^T$ is a random vector with covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Find $Var(X_1 + X_2 + X_3)$.

55. (16 pts) Suppose that (X, Y, Z) is a random vector with joint PDF $f_{X,Y,Z}$, where

$$f_{X,Y,Z}(x, y, z) = ce^{-(x^2 + 4xy + 5y^2)} z e^{-z} I_{(0,\infty)}(z)$$

for $(x, y, z) \in R^3$, and $c > 0$ is a constant.

(a) (2 pts) Find a version of the conditional PDF of Z given (X, Y).

(b) (6 pts) Find c, a PDF of Y, and a version of the conditional PDF of X given Y.

(c) (6 pts) Find $E(X|Y)$, $E(X^2|Y)$ and $Var(X|Y)$.

(d) (2 pts) Find the best linear predictor of X based on Y.

56. (4 pts) Suppose that X and Y are random variables such that $Y \sim N(0, 1)$ and a version of the conditional PDF of X given Y is $\{g_y : y \in R\}$, where g_y is the function $f_{\mu,\sigma}$ given in Problem 27 with $\mu = y$ and $\sigma = 1$. Find a version of the conditional PDF of Y given X.

57. (15 pts) Suppose that $\boldsymbol{X} = (X_1, X_2, X_3, X_4)^T$ is a random vector with $E(\boldsymbol{X}) = (0,0,0,0)^T$ and covariance matrix Σ, where

$$\Sigma = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Suppose that $\boldsymbol{X} \sim N(E(\boldsymbol{X}), \Sigma)$.

(a) (3 pts) Find the best linear predictor of $(X_1, X_4)^T$ based on X_2 and X_3.

(b) (3 pts) Find a version of the conditional PDF of X_1 given X_2.

(c) (3 pts) Find a version of the conditional PDF of X_1 given $(X_2, X_3, X_4)^T$.

(d) (3 pts) Find a version of the conditional PDF of $(X_1, X_2)^T$ given $(X_3, X_4)^T$.

(e) (3 pts) Find $E(X_1X_2|X_3, X_4)$.

58. (6 pts) Suppose that $\boldsymbol{Y}$, $\boldsymbol{X}$ and $\boldsymbol{U}$ are random vectors in R^m, R^n, R^m respectively such that $\boldsymbol{X}$ and $\boldsymbol{U}$ are independent and

$$\boldsymbol{Y} = g(\boldsymbol{X}) + \boldsymbol{U} \tag{4}$$

for some function $g : R^n \to R^m$. Suppse that g is differentiable, $\boldsymbol{X}$ has a PDF $f_{\boldsymbol{X}}$ that is positive on R^n, and $\boldsymbol{U}$ has a PDF $f_{\boldsymbol{U}}$ that is positive on R^m. For $\boldsymbol{x} \in R^n$, define

$$f_{\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}}(\boldsymbol{y}) = f_{\boldsymbol{U}}(\boldsymbol{y} - g(\boldsymbol{x})) \tag{5}$$

for $\boldsymbol{y} \in R^m$. Show that $\{f_{\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}} : \boldsymbol{x} \in R^n\}$ is a version of the conditional PDF of $\boldsymbol{Y}$ given $\boldsymbol{X}$.

59. (4 pts) For $\sigma_1 > 0$, $\sigma_2 > 0$, let Q_{σ_1,σ_2} denote the distribution of $(X + \varepsilon)$, where X and ε are two random variables such that $X \sim N(0, \sigma_1^2)$, $\varepsilon \sim N(0, \sigma_2^2)$, and X and ε are independent. Consider the family $\mathcal{C} = \{Q_{\sigma_1,\sigma_2} : (\sigma_1, \sigma_2) \in (0,\infty)\times(0,\infty)\}$. Determine whether $\mathcal{C}$ is identifiable and justify your answer.

60. (4 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$, where $\mu \in R$ and $\sigma > 0$ are unknown. Let $\bar{X} = \sum_{i=1}^n X_i/n$, $\bar{Y} = \sum_{i=1}^n X_i^2/n$, and $\bar{Z} = \sum_{i=1}^n (X_i - \mu)^2/n$. Which of the following statements are true? You may write down your answers directly without justification.

(a) $\bar{X}$ is a consistent estimator of μ.

(b) When $\mu = 0$, $\bar{Y}$ is a consistent estimator of σ^2.

(c) $\bar{Z}$ is a consistent estimator of σ^2.

(d) $\bar{Z}$ converges to σ^2 in probability as $n \to \infty$.

61. (4 pts) Suppose that $\theta \in R$ is a paramater to be estimated, and $\hat{\theta}$ is an estimator of θ such that

$$E(\hat{\theta}) = \theta$$

and

$$\lim_{n\to\infty} Var(\hat{\theta}) = 0.$$

Show that $\hat{\theta}$ is a consistent estimator of θ.

Hint: make use of Chebyshev's inequality.

62. (6 pts) Prove the following result.

Fact 4 Suppose that $\{X_{1,n}\}_{n=1}^{\infty}$, ..., $\{X_{k,n}\}_{n=1}^{\infty}$ are k sequences of random variables on $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is a σ-field on Ω and P is a probability function defined on $\mathcal{F}$, and $X_{1,n} \xrightarrow{\mathcal{P}} Y_1$, ..., $X_{k,n} \xrightarrow{\mathcal{P}} Y_k$ as $n \to \infty$ for some random vector $(Y_1, \ldots, Y_k)^T$ on $(\Omega, \mathcal{F}, P)$. Then

$$(X_{1,n}, \ldots, X_{k,n})^T \xrightarrow{\mathcal{P}} (Y_1, \ldots, Y_k)^T$$

as $n \to \infty$.

63. (6 pts) Suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of random vectors such that X_n converges to X almost surely as $n \to \infty$ for some random vector X that takes values in R^m. Suppose that a function g: $R^m \to R^k$ is continuous on R^m. Show that $g(X_n)$ converges to $g(X)$ almost surely as $n \to \infty$.

64. (4 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from $U(0, \theta)$, where $\theta > 0$. Find a consistent estimator of $1/\theta$ and justify your answer.

65. (12 pts) Recall that for $\alpha > 0$, $\Gamma(\alpha, 1)$ is the distribution of the random variable X in Problem 34. For $\alpha > 0$, $\beta > 0$, let $\Gamma(\alpha, \beta)$ denote the distribution of βY, where $Y \sim \Gamma(\alpha, 1)$. Suppose that $(X_1, \ldots, X_n)$ is a random sample from $\Gamma(\alpha, \beta)$, where $\alpha > 0$, $\beta > 0$.

   (a) (6 pts) Show that $E(X_1) = \alpha\beta$ and $Var(X_1) = \alpha\beta^2$.
   (b) (6 pts) Construct a consistent estimator of $(\alpha, \beta)^T$ based on $(X_1, \ldots, X_n)$ and justify your answer.

66. (10 pts) Suppose that $(X_1, \ldots, X_n)$ is a random sample from a discrete distribution with three possible values a_1, a_2, a_3. For $j = 1, 2, 3$, let

$$p_j = P(X_1 = a_j),$$

$$\hat{p}_j = \frac{1}{n}\sum_{i=1}^{n} I_{\{a_j\}}(X_i),$$

and

$$Y_{n,j} = \sqrt{n}(\hat{p}_j - p_j).$$

Let $Y_n = (Y_{n,1}, Y_{n,2}, Y_{n,3})^T$.

   (a) (4 pts) Find the limiting distribution of Y_n. That is, find the distribution D_0 such that Y_n converges to D_0 in distribution as $n \to \infty$.
   (b) (6 pts) Let

$$A = \begin{pmatrix} \sqrt{p_1} & 0 & 0 \\ 0 & \sqrt{p_2} & 0 \\ 0 & 0 & \sqrt{p_3} \end{pmatrix}.$$

   Show that the limiting distribution of $A^{-1}Y_n$ is $N(\mathbf{0}, \Sigma)$ using the continuous mapping theorem for convergence in distribution (given in Page 6 of the handout "Estimation and types of convergence"), where the covariance matrix Σ satisfies $\Sigma^2 = \Sigma$.

Remark. From the result of this problem and Fact 4 in the handout "Estimation and types of convergence", the limiting distribution of the test statistic of the chi-squared goodness of fit test is a χ^2 distribution under H_0 when $m = 3$.