Range of a transformed variable

- Suppose that X is a random variable such that $P(X \in S_X) = 1$, then S_X can be viewed as a range of X. Suppose that $Y = g(X)$ and g is a one-to-one function, then a range of Y is $S_Y = \{g(x) : x \in S_X\}$. Below we consider some conditions under which S_Y can be easily found.

- Case 1. Suppose that $g'(x) > 0$ for $x \in S_X = (a, b)$, and both $L_1 = \lim_{x\to a^+} g(x)$ and $L_2 = \lim_{x\to b^-} g(x)$ can be defined, then $S_Y = (L_1, L_2)$.
  - If $a = -\infty$, then $L_1 = \lim_{x\to a^+} g(x)$ should be replaced by $L_1 = \lim_{x\to -\infty} g(x)$.
  - If $b = \infty$, then $L_2 = \lim_{x\to b^-} g(x)$ should be replaced by $L_2 = \lim_{x\to\infty} g(x)$.

- Case 2. Suppose that $g'(x) < 0$ for $x \in S_X = (a, b)$, and both $L_1 = \lim_{x\to a^+} g(x)$ and $L_2 = \lim_{x\to b^-} g(x)$ can be defined, then $S_Y = (L_2, L_1)$.
  - If $a = -\infty$, then $L_1 = \lim_{x\to a^+} g(x)$ should be replaced by $L_1 = \lim_{x\to -\infty} g(x)$.
  - If $b = \infty$, then $L_2 = \lim_{x\to b^-} g(x)$ should be replaced by $L_2 = \lim_{x\to\infty} g(x)$.

- If $g'(x)$ does not exist or $g'(x) = 0$ for some $x \in (a, b)$, then divide the interval (a, b) into disjoint sub-intervals using the points at which $g' = 0$ or g' does not exist as break points.

- If S_X is a union of k disjoint open intervals $I_1, \ldots, I_k$, then $S_Y = \cup_{i=1}^{k} \{g(x) : x \in I_i\}$.

- Example 1. Suppose that $P(X \in (-1, 1)) = 1$ and $Y = X/(1 + X)$. Find $S_Y = \{x/(1 + x) : x \in (-1, 1)\}$.

  Ans. $S_Y = (-\infty, 1/2)$.

- Example 2. Suppose that $P(X \in (0, \infty)) = 1$ and $Y = 1 - e^{-X}$. Find $S_Y = \{1 - e^{-x} : x \in (0, \infty)\}$.

  Ans. $S_Y = (0, 1)$.

- Example 3. Suppose that $P(X \in (0, 1) \cup (1, \infty)) = 1$ and $Y = g(X)$, where

$$g(x) = \begin{cases} x & \text{if } x \in (0, 1); \\ e^x & \text{if } x \in (1, \infty). \end{cases}$$

  Find $S_Y = \{g(x) : x \in (0, 1) \cup (1, \infty)\}$.

  Ans. $S_Y = (0, 1) \cup (e, \infty)$.

- Example 4. Suppose that $P(X \in (0,\infty)) = 1$ and $Y = g(X)$, where

$$g(x) = \begin{cases} x & \text{if } x \in (0,1]; \\ 1 + 1/x & \text{if } x \in (1,\infty). \end{cases}$$

  Find $S_Y = \{g(x) : x \in (0,\infty)\}$.

  Ans. $S_Y = (0,2)$.

- It can be helpful to plot the graph of $y = g(x)$ for $x \in S_X$ using software like R. The R command for plot $y = g(x)$ for $x \in (a,b)$ is `curve(g,a,b)`. R can be downloaded from its official website at

  `https://cran.r-project.org`

- Example 5. Consider the function g in Example 4. Plot the graph of $y = g(x)$ for $x \in (0,4)$ using R.

  Sol. Running the following R codes gives the plot required.

```
g <- function(x){
 ans1 <- x
 ans2 <- 1+1/x
 ans1[x>1] <- ans2[x>1]
 return(ans1)
}
curve(g, 0, 4)
```