Open sets in R^n

- Definition of an open ball in R^n. For $\boldsymbol{a} \in R^n$ and $\delta > 0$, the open ball centered at $\boldsymbol{a}$ with radius δ is the set
$$\{\boldsymbol{x} \in R^n : \|\boldsymbol{x} - \boldsymbol{a}\| < \delta\},$$
where for $\boldsymbol{x} = (x_1, \ldots, x_n) \in R^n$ and $\boldsymbol{a} = (a_1, \ldots, a_n) \in R^n$,
$$\|\boldsymbol{x} - \boldsymbol{a}\| = \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2}$$
is the Euclidean distance between $\boldsymbol{x}$ and $\boldsymbol{a}$.

- We will use $B(\boldsymbol{a}, \delta)$ to denote the open ball centered at $\boldsymbol{a}$ with radius δ.

- Definition of an interior point (內點) of a set in R^n. Suppose that $S \subset R^n$ and $\boldsymbol{a} \in S$. If there exists some $\delta > 0$ such that $B(\boldsymbol{a}, \delta) \subset S$, then $\boldsymbol{a}$ is called an interior point of S.
  - Example. Every point in the open interval $(0, 1)$ is an interior point of $(0, 1)$.

- Definition of a boundary point (邊界點) of a set in R^n. Suppose that $S \subset R^n$ and $\boldsymbol{a} \in R^n$. If for every $\delta > 0$, $B(\boldsymbol{a}, \delta)$ contains some point outside S and some point in S, then $\boldsymbol{a}$ is called a boundary point of S. Note.
  - A boundary point of S may or may not belong to S. For instance, both 0 and 1 are boundary points of the interval $[0, 1)$, yet $0 \in [0, 1)$ and $1 \notin [0, 1)$.

- Definition of an open set in R^n. For $S \subset R^n$, S is called an open set in R^n if and only if every point in S is an interior point of S.
  - Note that an open set does not contain any of its boundary points by definition.

- Example 1. An open ball in R^n is an open set in R^n.

  To see that an open ball in R^n is an open set in R^n, consider the open ball $B(\boldsymbol{a}, \delta)$, where $\boldsymbol{a} \in R^n$ and $\delta > 0$. For $\boldsymbol{x} \in B(\boldsymbol{a}, \delta)$, we will show that $\boldsymbol{x}$ is an interior point of $B(\boldsymbol{a}, \delta)$. Take
$$\delta_1 = \delta - \|\boldsymbol{x} - \boldsymbol{a}\|,$$
then $\delta_1 > 0$ and for $\boldsymbol{y} \in B(\boldsymbol{x}, \delta_1)$,
$$\|\boldsymbol{y} - \boldsymbol{a}\| \leq \|\boldsymbol{y} - \boldsymbol{x}\| + \|\boldsymbol{x} - \boldsymbol{a}\| < \delta_1 + \|\boldsymbol{x} - \boldsymbol{a}\| = \delta,$$
so $\boldsymbol{y} \in B(\boldsymbol{a}, \delta)$. Therefore, $B(\boldsymbol{x}, \delta_1) \subset B(\boldsymbol{a}, \delta)$ and $\boldsymbol{x}$ is an interior point of $B(\boldsymbol{a}, \delta)$. Since for every $\boldsymbol{x} \in B(\boldsymbol{a}, \delta)$, $\boldsymbol{x}$ is an interior point of $B(\boldsymbol{a}, \delta)$, the open ball $B(\boldsymbol{a}, \delta)$ is an open set in R^n.

- To determine whether a set in R^n contains a nonempty open set in R^n, the following result is useful.

  Fact 1 *Suppose that $S \subset R^n$. Then*

  $$S \textit{ contains a nonempty open set in } R^n \ \Leftrightarrow\ S \textit{ contains an open ball in } R^n$$

  Proof of Fact 1. Since an open ball in R^n is a nonempty open set in R^n, the "$\Leftarrow$" direction holds true clearly. We only need to prove the "$\Rightarrow$" direction. Suppose that S contains a nonempty open set B in R^n. Let $\boldsymbol{x}$ be a point in B, then $\boldsymbol{x}$ is an interior point of B since B is open. Therefore, there exists $\delta > 0$ such that $B(\boldsymbol{x}, \delta) \subset B \subset S$, and S contains the open ball $B(\boldsymbol{x}, \delta)$. The proof of the "$\Rightarrow$" direction is complete.