Method of moment estimators

- Suppose that $(X_1, \ldots, X_n)$ is a random sample and X_1 has a PDF or PMF f_θ, where θ is in some parameter space Θ. One way to find an estimator of θ is to use the method of moments. Suppose that X_1 is a random variable and θ is k-dimensional. Compute $E(X_1^r)$ for $r = 1, \ldots, k$ and we have

$$\begin{cases} E(X_1) = g_1(\theta) \\ E(X_1^2) = g_2(\theta) \\ \vdots \\ E(X_1^k) = g_k(\theta) \end{cases} \tag{1}$$

  for some functions $g_1, \ldots, g_k$. Replace $E(X_1^r)$ with $\sum_{i=1}^n X_i^r/n$ for $r = 1, \ldots, k$ in (1) and solve for θ. Then the solution gives an estimator of θ, which is called a method of moment estimator of θ.

- If (1) defines a function g so that

$$\theta = g(E(X_1), E(X_1^2), \ldots, E(X_1^k))$$

  and g is continuous at $(E(X_1), E(X_1^2), \ldots, E(X_1^k))$, then

$$g\left(\frac{\sum_{i=1}^n X_i}{n}, \frac{\sum_{i=1}^n X_i^2}{n}, \ldots, \frac{\sum_{i=1}^n X_i^k}{n}\right)$$

  is a consistent estimator of θ.

- Example 1. Suppose that $(X_1, \ldots, X_n)$ is a random sample and X_i has a PDF $f_{a,b}$, where $a < b$ and

$$f_{a,b}(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b; \\ 0 & \text{otherwise.} \end{cases}$$

  Find a method of moment estimator of the parameter vector (a, b).

  A sketch of solution. Compute $E(X_1)$ and $E(X_1^2)$ and we have

$$\begin{cases} E(X_1) = \frac{a+b}{2} \\ E(X_1^2) = \left(\frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12} \end{cases}$$

  Let $\overline{X} = \sum_{i=1}^n X_i/n$ and $\overline{X^2} = \sum_{i=1}^n X_i^2/n$. Solving

$$\begin{cases} \overline{X} = \frac{a+b}{2} \\ \overline{X^2} = \left(\frac{a+b}{2}\right)^2 + \frac{(b-a)^2}{12} \end{cases}$$

  for a, b and we have

$$a = \overline{X} - \sqrt{3(\overline{X^2} - (\overline{X})^2)}$$

and

$$b = \overline{X} + \sqrt{3(\overline{X^2} - (\overline{X})^2)}.$$

A method of moment estimator of (a, b) is

$$\left(\overline{X} - \sqrt{3(\overline{X^2} - (\overline{X})^2)}, \overline{X} + \sqrt{3(\overline{X^2} - (\overline{X})^2)}\right).$$