Probability calculation based on MGF

- Definition of a characteristic function. Suppose that $(X_1, \ldots, X_k)$ is a random vector. For $(t_1, \ldots, t_k) \in R^k$, define

$$\varphi(t_1, \ldots, t_k) = E\left(e^{i(t_1X_1+\cdots+t_kX_k)}\right),$$

where $i = \sqrt{-1}$, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ for $\theta \in (-\infty, \infty)$, and

$$E(U + iV) = E(U) + iE(V)$$

for a random vector (U, V). Then φ is called the characteristic function of $(X_1, \ldots, X_k)$. If $(X_1, \ldots, X_k)$ has MGF $M_{X_1,\ldots,X_k}$, then

$$\varphi(t_1, \ldots, t_k) = M_{X_1,\ldots,X_k}(it_1, \ldots, it_k)$$

for $(t_1, \ldots, t_k) \in R^k$.

- **Fact 1** *Suppose that $(X_1, \ldots, X_k)$ is a random vector with characteristic function φ, then*

$$\begin{aligned} &P((X_1, \ldots, X_k) \in (a_1, b_1] \times \cdots \times (a_k, b_k]) \\ &= \lim_{T\to\infty} \frac{1}{(2\pi)^k} \int_{[-T,T]^k} h(t_1, \ldots, t_k) d(t_1, \ldots, t_k), \end{aligned} \tag{1}$$

where

$$h(t_1, \ldots, t_k) = \prod_{j=1}^{k} \frac{e^{-it_ja_j} - e^{-it_jb_j}}{it_j} \varphi(t_1, \ldots, t_k).$$

The Proof of Fact 1 can be found in Billingsley [1].

- Example 1. For $\alpha > 0$ and $x \in (-\infty, \infty)$, let

$$f_\alpha(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} & \text{if } x > 0; \\ 0 & \text{if } x \le 0, \end{cases} \tag{2}$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$. Suppose that X has PDF f_α, then from (18) in the solution to Problem 42,

$$E(e^{tX}) = (1 - t)^{-\alpha}$$

for $t < 1$. Find an expression for $P(a < X \le b)$ using (1) for a, $b \in (-\infty, \infty)$, $a < b$.

Sol. Let

$$h(t) = \left(\frac{e^{-ita} - e^{-itb}}{it}\right)(1 - it)^{-\alpha}$$

for $t \in (-\infty, \infty)$, then by (1),

$$P(X \in (a, b]) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} h(t) dt. \tag{3}$$

- Computing probabilities using characteristic functions can be done using the software R.

  – The R command for computing $\int_a^b f(x)dx$ is

    ```
    integrate(f,a,b)$value
    ```

    where a, b can be $-\infty$, ∞ respectively.

  – The R command for creating a complex vector is `complex`. For example, running the command

    ```
    z <- complex(real=c(4,2,1), imaginary=c(3,6,5))
    ```

    creates the vector $z = (4+3i, 2+6i, 1+5i)$ in R, and running the commands

    ```
    a <- pi/2
    b <- pi
    z <- complex(modulus=c(4,2), argument=c(a,b))
    ```

    creates the vector $z = (4e^{ia}, 2e^{ib})$ in R for $(a, b) = (\pi/2, \pi)$.

  – The R commands for getting the real part and the imaginary part of a complex vector `z` are `Re(z)` and `Im(z)` respectively:

    ```
    z <- complex(real=c(4,2,1), imaginary=c(3,6,5))
    Re(z); Im(z)
    ```

- Example 2. Compute $P(X \in (1, 2])$ for the X in Example 1 with $\alpha = 2$ using

  (a) Equation (3) and

  (b) the PDF f_α in (2)

  respectively.

  Sol. We will compute the probability using the software R.

```
#(a)
a <- 1; b <- 2;alpha <- 2
h.real <- function(s){
 n <- length(s)
 v1 <- rep(1,n)
 v0 <- rep(0,n)
 za <- complex(modulus = v1, argument = -a*s)
 zb <- complex(modulus = v1, argument = -b*s)
 z <- (za-zb)/complex(real=v0, imaginary=s)
 ans <- z*complex(real=v1, imaginary=-s)^(-alpha)
 return(Re(ans))
}
```

```
integrate(h.real,-Inf,Inf)$value/(2*pi) #0.3297519



#(b)
alpha <- 2
f <- function(x){ x^(alpha-1)*exp(-x)  }
integrate(f,0, Inf)$value  ### The integral is 1, so f is the PDF f_alpha.
integrate(f,1,2)$value     ##P(X in (1,2]) = 0.329753
```

Reference

[1] P. Billingsley, *Probability and measure*, John Wiley & Sons, New York, 2nd ed., 1986.