Finding a joint PDF using conditional and marginal PDFs

- Suppose that $(\boldsymbol{X}^T, \boldsymbol{Y}^T)$ is a random vector, $\boldsymbol{Y}$ takes values in R^k and $f_{\boldsymbol{X}}$ is a PDF of $\boldsymbol{X}$. Suppose that a version of the conditional PDF of $\boldsymbol{Y}$ given $\boldsymbol{X} = \boldsymbol{x}$ is $f_{\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}}$ for $\boldsymbol{x} \in \{\boldsymbol{x} : f_{\boldsymbol{X}}(\boldsymbol{x}) > 0\}$. Let

$$f_{\boldsymbol{X},\boldsymbol{Y}}(\boldsymbol{x}, \boldsymbol{y}) = f_{\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}}(\boldsymbol{y}) f_{\boldsymbol{X}}(\boldsymbol{x})$$

  for $\boldsymbol{y} \in R^k$ and $\boldsymbol{x} \in \{\boldsymbol{x} : f_{\boldsymbol{X}}(\boldsymbol{x}) > 0\}$. Then $f_{\boldsymbol{X},\boldsymbol{Y}}$ is a joint PDF of $(\boldsymbol{X}^T, \boldsymbol{Y}^T)$.

- Example 1. For $(y_1, y_2, y_3) \in R^3$, define

$$g_{y_2,y_3}(y_1) = \frac{1}{\sqrt{2\pi}} e^{-(y_1-y_2-y_3)^2/2}$$

  and

$$h(y_2, y_3) = \left(\frac{\sqrt{3}}{2\pi}\right) e^{-(2y_2^2+2y_3^2+2y_2y_3)/2}.$$

  Suppose that (Y_1, Y_2, Y_3) is a random vector, h is a PDF of (Y_2, Y_3) and g_{y_2,y_3} is a version of the conditional PDF of Y_1 given $(Y_2, Y_3) = (y_2, y_3)$ for $(y_2, y_3) \in R^2$. Find a joint PDF of (Y_1, Y_2, Y_3) and show that the distribution of (Y_1, Y_2, Y_3) is a multivariate normal distribution.

  Sol. Let $f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = g_{y_2,y_3}(y_1)h(y_2, y_3)$ for $(y_1, y_2, y_3) \in R^3$, then f_{Y_1,Y_2,Y_3} is a joint PDF of (Y_1, Y_2, Y_3). It can be shown that for $(y_1, y_2, y_3) \in R^3$,

$$f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \det(\Sigma)^{-1/2} e^{-0.5\boldsymbol{y}^T\Sigma^{-1}\boldsymbol{y}},$$

  where $\boldsymbol{y} = (y_1, y_2, y_3)^T$ and Σ is the covariance matrix of $(Y_1, Y_2, Y_3)^T$. Thus the distribution of $(Y_1, Y_2, Y_3)^T$ is $N(\boldsymbol{\mu}, \Sigma)$ with $\boldsymbol{\mu} = (0, 0, 0)^T$.