Independence of two random vectors

- Definition. Suppose that $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$ are random vectors. X and Y are independent means

  $$P(X \in A_1 \text{ and } Y \in A_2) = P(X \in A_1)P(Y \in A_2)$$

  for all $A_1 \subset R^m$ and $A_2 \subset R^n$.

- Note. Suppose that X and Y are random vectors and X and Y are independent. Then $u(X)$ and $v(Y)$ are independent for all functions u and v.

- CDF case.

  Fact 1 Suppose that $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_m)$ are random vectors. X and Y are independent if and only if

  $$F_{X_1,\ldots,X_m,Y_1,\ldots,Y_n}(x_1, \ldots, x_m, y_1, \ldots, y_n) = F_{X_1,\ldots,X_m}(x_1, \ldots, x_m)F_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n)$$

  for all $(x_1, \ldots, x_m) \in R^m$, $(y_1, \ldots, y_n) \in R^n$.

- PDF case.

  Fact 2 Suppose that $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n$ are random vectors and (X, Y) has a joint PDF. Suppose that X and Y have PDFs f_X and f_Y respectively. Define

  $$g(x_1, \ldots, x_m, y_1, \ldots, y_n) = f_X(x_1, \ldots, x_m)f_Y(y_1, \ldots, y_n)$$

  for $(x_1, \ldots, x_m) \in R^m$ and $(y_1, \ldots, y_n) \in R^n$, Then g can be a PDF of (X, Y) if and only if X and Y are independent.

- PMF case.

  Fact 3 Suppose that $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$ are discrete random vectors and (X, Y) has a joint PMF $p_{X,Y}$. Let p_X and p_Y be the PMFs of X and Y respectively. Then X and Y are independent if and only if

  $$p_{X,Y}(x_1, \ldots, x_m, y_1, \ldots, y_n) = p_X(x_1, \ldots, x_m)p_Y(y_1, \ldots, y_n)$$

  for $(x_1, \ldots, x_m) \in R^m$ and $(y_1, \ldots, y_n) \in R^n$.

- General case.

  Fact 4 Suppose that $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$ are random vectors. X and Y are independent if and only if $< a, X >$ and $< b, Y >$ are independent for all $a \in R^m$, $b \in R^n$.

For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, the notation $< x, y >$ means $\sum_{i=1}^n x_i y_i$, the inner product of x and y.

- MGF case.

  Fact 5 Suppose that $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$ are random vectors with joint MGF $M_{X,Y}$ and $M_{X,Y}(s_1, \ldots, s_m, t_1, \ldots, t_n) < \infty$ for $(s_1, \ldots, s_m)$ and $(t_1, \ldots, t_n)$ such that $\max\{|s_1|, \ldots, |s_m|, |t_1|, \ldots, |t_n|\} < \delta$ for some $\delta > 0$. Then X and Y are independent if and only if

  $$M_{X,Y}(s_1, \ldots, s_m, t_1, \ldots, t_n) = M_X(s_1, \ldots, s_m) M_Y(t_1, \ldots, t_n)$$

  for $(s_1, \ldots, s_m)$ and $(t_1, \ldots, t_n)$ such that $\max\{|s_1|, \ldots, |s_m|, |t_1|, \ldots, |t_n|\} < \delta$, where M_X and M_Y are the MGFs of X and Y respectively.

  Note that

  $$M_X(s_1, \ldots, s_m) = M_{X,Y}(s_1, \ldots, s_m, 0, \ldots, 0)$$

  for $(s_1, \ldots, s_m) \in R^m$ and

  $$M_Y(t_1, \ldots, t_n) = M_{X,Y}(0, \ldots, 0, t_1, \ldots, t_n)$$

  for $(t_1, \ldots, t_n) \in R^n$.

- Example 1. Suppose that X_1, X_2, X_3 are independent random variables and the distribution of each X_i is $N(\mu, \sigma^2)$, where μ and σ are constants and $\sigma > 0$. Let $\bar{X} = (X_1 + X_2 + X_3)/3$ and $Y = (X_1 - \bar{X}, X_2 - \bar{X}, X_3 - \bar{X})^T$. Show that $\bar{X}$ and Y are independent.

  Sol. We will first find the joint MGF of $\bar{X}$ and Y. From Example 4(a) in the handout "Independent random variables",

  $$E(e^{wX_1}) = e^{\mu w + 0.5\sigma^2 w^2}$$

  for $w \in R$. Let $M(w) = E(e^{wX_1})$ for $w \in R$. For $s \in R$ and $t = (t_1, t_2, t_3)^T \in R^3$,

  $$\begin{aligned} E(e^{s\bar{X} + t^T Y}) &= E \exp\left(s\bar{X} + \sum_{i=1}^3 t_i (X_i - \bar{X}) \right) \\ &= E \exp\left(\sum_{i=1}^3 t_i X_i + (s - (t_1 + t_2 + t_3))\bar{X} \right) \\ &= E e^{a_1 X_1 + a_2 X_2 + a_3 X_3}, \end{aligned}$$

  where $a_i = t_i + (s - (t_1 + t_2 + t_3))/3$ for i = 1,2,3. Since X_1, X_2, X_3 are independent and have the same distribution $N(\mu, \sigma^2)$, we have

  $$E(e^{s\bar{X} + t^T Y}) = M(a_1) M(a_2) M(a_3)$$

$$
\begin{aligned}
&= \prod_{i=1}^{3} M(t_i + (s - (t_1 + t_2 + t_3))/3) \\
&= \exp\left(\sum_{i=1}^{3}\left[\mu\left(t_i + \frac{s-3\bar{t}}{3}\right) + 0.5\sigma^2\left(t_i + \frac{s-3\bar{t}}{3}\right)^2\right]\right),
\end{aligned}
$$

where $\bar{t} = (t_1 + t_2 + t_3)/3$. It can be shown that

$$
\begin{aligned}
&\sum_{i=1}^{3}\left[\mu\left(t_i + \frac{s-3\bar{t}}{3}\right) + 0.5\sigma^2\left(t_i + \frac{s-3\bar{t}}{3}\right)^2\right] \\
&= \mu \cdot s + \frac{\sigma^2 s^2}{6} + 0.5\sigma^2 \sum_{i=1}^{3}(t_i - \bar{t})^2, \qquad (1)
\end{aligned}
$$

where $\bar{t} = (t_1 + t_2 + t_3)/3$. Therefore, for $s \in R$ and $t = (t_1, t_2, t_3)^T \in R^3$,

$$
E(e^{s\bar{X} + t^T Y}) = M_1(s) M_2(t),
$$

where

$$
M_1(s) = \exp\left(\mu \cdot s + \frac{\sigma^2 s^2}{6}\right)
$$

and

$$
\begin{aligned}
M_2(t) &= \exp\left(0.5\sigma^2 \sum_{i=1}^{3}(t_i - \bar{t})^2\right) \\
&= \exp\left(0.5\sigma^2 \sum_{i=1}^{3}(t_i - (t_1 + t_2 + t_3)/3)^2\right).
\end{aligned}
$$

It is clear that

$$
E(e^{s\bar{X} + t^T Y})\Big|_{t=(0,0,0)^T} = M_1(s)
$$

and

$$
E(e^{s\bar{X} + t^T Y})\Big|_{s=0} = M_2(t),
$$

so

$$
E(e^{s\bar{X} + t^T Y}) = \left(E(e^{s\bar{X} + t^T Y})\Big|_{t=(0,0,0)^T}\right)\left(E(e^{s\bar{X} + t^T Y})\Big|_{s=0}\right)
$$

for $s \in R$, $t \in R^3$, which implies that $\bar{X}$ and Y are independent.

- IID= Indepenent and Identically Distributed (獨立同分布). Suppose that random variables $X_1, \ldots, X_n$ are indepenent and identically distributed, then we say that $X_1, \ldots, X_n$ are IID.
  - In Example 1, X_1, X_2, X_3 are IID.