Distribution of a transformed random vector with PDF

- Suppose that $(X_1, \ldots, X_k)$ is a random vector with joint PDF $f_{X_1,\ldots,X_k}$. Let $S_X = \{x \in R^k : f_{X_1,\ldots,X_k}(x) > 0\}$. Suppose that $y_1, \ldots, y_k$ are differentiable functions on S_X with continuous partial derivatives, and the transform

$$g(x) = (y_1(x), \ldots, y_k(x))$$

for $x \in S_X$ is one-to-one. Let $S_Y = \{g(x) : x \in S_X\}$ and let $(x_1(y), \ldots, x_k(y)) = g^{-1}(y)$ for $y \in S_Y$. Let D be the $k \times k$ matrix whose (i, j)-th element is

$$\frac{\partial x_i(y)}{\partial y_j}$$

and define

$$f_Y(y) = \begin{cases} f_{X_1,\ldots,X_k}(x_1(y), \ldots, x_k(y))|\text{determinant}(D)| & \text{if } y \in S_Y; \\ 0 & \text{otherwise,} \end{cases}$$

then f_Y is a PDF of $(y_1(X_1, \ldots, X_k), \ldots, y_k(X_1, \ldots, X_k))$.

- Example 1. Suppose that X_1, X_2, X_3 are independent random variables and the distribution of each X_i is $N(0, 1)$. Let $\bar{X} = (X_1 + X_2 + X_3)/3$ and $(Y_1, Y_2, Y_3) = (\bar{X}, X_1 - \bar{X}, X_2 - \bar{X})$. Find a version of the conditional PDF of Y_1 given $(Y_2, Y_3) = (y_2, y_3)$ for $(y_2, y_3) \in R^2$ based on the joint PDF of (Y_1, Y_2, Y_3).

Sol. Given (y_1, y_2, y_3), solving for (x_1, x_2, x_3) such that

$$(y_1, y_2, y_3) = ((x_1+x_2+x_3)/3, x_1-(x_1+x_2+x_3)/3, x_2-(x_1+x_2+x_3)/3)$$

gives $x_1 = y_1 + y_2$, $x_2 = y_1 + y_3$ and $x_3 = 3y_1 - (y_1 + y_2 + y_3 + y_1) = y_1 - y_2 - y_3$. Let

$$D = \begin{pmatrix} \frac{\partial(y_1+y_2)}{\partial y_1} & \frac{\partial(y_1+y_2)}{\partial y_2} & \frac{\partial(y_1+y_2)}{\partial y_3} \\ \frac{\partial(y_1+y_3)}{\partial y_1} & \frac{\partial(y_1+y_3)}{\partial y_2} & \frac{\partial(y_1+y_3)}{\partial y_3} \\ \frac{\partial(y_1-y_2-y_3)}{\partial y_1} & \frac{\partial(y_1-y_2-y_3)}{\partial y_2} & \frac{\partial(y_1-y_2-y_3)}{\partial y_3} \end{pmatrix},$$

then

$$D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix},$$

and

$$\begin{aligned} &\text{determinant}(D) \\ &= 1 \cdot \text{determinant}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} - 1 \cdot \text{determinant}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + 0 \cdot \text{determinant}\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \\ &= 3. \end{aligned}$$

Let

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \frac{1}{\sqrt{2\pi}}e^{-x_1^2/2}\left(\frac{1}{\sqrt{2\pi}}e^{-x_2^2/2}\right)\left(\frac{1}{\sqrt{2\pi}}e^{-x_3^2/2}\right)$$

for $(x_1,x_2,x_3) \in R^3$, then f_{X_1,X_2,X_3} is a joint PDF of (X_1,X_2,X_3). Let

$$\begin{aligned} &f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) \\ &= f_{X_1,X_2,X_3}(y_1+y_2, y_1+y_3, y_1-y_2-y_3)\cdot|3| \end{aligned}$$

for $(y_1,y_2,y_3) \in R^3$, then f_{Y_1,Y_2,Y_3} is a PDF of (Y_1,Y_2,Y_3). Simplify the expression of f_{Y_1,Y_2,Y_3} and we have

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = g(y_1)h(y_2,y_3),$$

where

$$g(y_1) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-y_1^2/(2\sigma^2)}$$

with $\sigma^2 = 1/3$ and

$$h(y_2,y_3) = \left(\frac{\sqrt{3}}{2\pi}\right)e^{-(2y_2^2+2y_3^2+2y_2y_3)/2}$$

for $(y_1,y_2,y_3) \in R^3$.

It can be shown that g is a PDF of Y_1 and Y_1 and (Y_2,Y_3) are independent. Thus g is a version of the conditional PDF of Y_1 given $(Y_2,Y_3) = (y_2,y_3)$ for $(y_2,y_3) \in R^2$.