Distribution of a transformed random vector

- Suppose that (X, Y) is a vector of two discrete random variables with joint PMF $p_{X,Y}$. Then for $(U, V) = (u(X, Y), v(X, Y))$,

$$P((U, V) = (u_0, v_0)) = \sum_{(x,y):p_{X,Y}(x,y)>0 \text{ and } (u(x,y),v(x,y))=(u_0,v_0)} p_{X,Y}(x, y)$$

can be computed directly for a given pair (u_0, v_0), so the joint PMF of (U, V) can be obtained.

- Example 1. Suppose that (X, Y) is a vector of discrete random variables with joint PMF $p_{X,Y}$, where

$$p_{X,Y}(x, y) = \begin{cases} 0.5 & \text{if } (x, y) = (1, 2); \\ 0.3 & \text{if } (x, y) = (3, 4); \\ 0.1 & \text{if } (x, y) = (3, 6); \\ 0.1 & \text{if } (x, y) = (3, 7); \\ 0 & \text{otherwise.} \end{cases}$$

Find the joint PMF of $(U, V) = (X + Y, X - Y)$.

Sol. The possible values of (X, Y) and $(U, V) = (X + Y, X - Y)$ and the corresponding probabilities are listed as follows.

(x, y)	$(x + y, x - y)$	$P((X, Y) = (x, y))$
$(1, 2)$	$(3, -1)$	0.5
$(3, 4)$	$(7, -1)$	0.3
$(3, 6)$	$(9, -3)$	0.1
$(3, 7)$	$(10, -4)$	0.1

From the above table, the possible values of (U, V) are $(3, -1)$, $(7, -1)$, $(9, -3)$ and $(10, -4)$. Let $p_{U,V}$ be the PMF of (U, V), then

$$p_{U,V}(u, v) = \begin{cases} 0.5 & \text{if } (u, v) = (3, -1); \\ 0.3 & \text{if } (u, v) = (7, -1); \\ 0.1 & \text{if } (u, v) = (9, -3); \\ 0.1 & \text{if } (u, v) = (10, -4); \\ 0 & \text{otherwise.} \end{cases}$$

- **Fact 1** *Suppose that (X, Y) has joint PDF $f_{X,Y}$ and $S_{X,Y} = \{(x, y) : f_{X,Y}(x, y) > 0\}$. Suppose that $H(x, y) = (u(x, y), v(x, y))$ for $(x, y) \in S_{X,Y}$ such that H is a one-to-one function on $S_{X,Y}$ and u, v have continuous partial derivatives on $S_{X,Y}$. Let $(U, V) = H(X, Y) = (u(X, Y), v(X, Y))$ and*

$$T = \{(u(x, y), v(x, y)) : (x, y) \in S_{X,Y}\}.$$

For $(u,v) \in T$, *let* $(x(u,v), y(u,v)) = H^{-1}(u,v)$. *Let*

$$J(u,v) = \text{ determinant of } \begin{pmatrix} x_u(u,v) & x_v(u,v) \\ y_u(u,v) & y_v(u,v) \end{pmatrix}$$

for $(u,v) \in T$. *Define*

$$f_{U,V}(u,v) = \begin{cases} f_{X,Y}(x(u,v), y(u,v))|J(u,v)| & \text{if } (u,v) \in T; \\ 0 & \text{otherwise,} \end{cases}$$

then $f_{U,V}$ *is a PDF of* (U,V). *Here* $J(u,v)$ *can also be computed using*

$$\begin{pmatrix} u_x(x(u,v), y(u,v)) & u_y(x(u,v), y(u,v)) \\ v_x(x(u,v), y(u,v)) & v_y(x(u,v), y(u,v)) \end{pmatrix} \begin{pmatrix} x_u(u,v) & x_v(u,v) \\ y_u(u,v) & y_v(u,v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Example 2. Suppose that (X,Y) has joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x,y) = \begin{cases} e^{-x}e^{-y} & \text{if } x > 0 \text{ and } y > 0; \\ 0 & \text{otherwise} \end{cases}$$

Find a PDF of $(U,V) = (X+Y, X/(X+Y))$.
Sol. Solving $(u,v) = (x+y, x/(x+y))$ for (x,y) gives $(x,y) = (uv, u-uv)$. Compute

$$\begin{aligned} J(u,v) &= \text{determinant of } \begin{pmatrix} \frac{\partial}{\partial u}(uv) & \frac{\partial}{\partial v}(uv) \\ \frac{\partial}{\partial u}(u-uv) & \frac{\partial}{\partial v}(u-uv) \end{pmatrix} \\ &= \text{determinant of } \begin{pmatrix} v & u \\ 1-v & -u \end{pmatrix} \\ &= v(-u) - u(1-v) = -u. \end{aligned}$$

Let

$$T = \{(u(x,y), v(x,y)) : f_{X,Y}(x,y) > 0\} = \{(x+y, x/(x+y)) : x > 0 \text{ and } y > 0\}$$

and define

$$f_{U,V}(u,v) = \begin{cases} f_{X,Y}(uv, u-uv)|-u| & \text{if } (u,v) \in T; \\ 0 & \text{otherwise,} \end{cases}$$

then $f_{U,V}$ is a PDF of (U,V). The expression of $f_{U,V}$ can be simplified as follows:

$$f_{U,V}(u,v) = \begin{cases} ue^{-u} & \text{if } (u,v) \in T; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$T = \{(u,v) : u > 0, v > 0, uv > 0 \text{ and } u - uv > 0\} = (0,\infty) \times (0,1).$$