Expectation of $g(X, Y)$

- Suppose that (X, Y) is a vector of two random variables defined on the same sample space.

  – If X, Y are both discrete, then

$$E(g(X,Y)) = \sum_{(x,y):p_{X,Y}(x,y)>0} g(x,y)p_{X,Y}(x,y). \tag{1}$$

  – If (X, Y) has joint PDF $f_{X,Y}$, then

$$E(g(X,Y)) = \int_{R^2} g(x,y)f_{X,Y}(x,y)d(x,y). \tag{2}$$

- Fact 1 Suppose that (X, Y) is a vector of two random variables such that $E(X)$ and $E(Y)$ are finite. Then

$$E(X + Y) = E(X) + E(Y).$$

  The proof of Fact 1 is based on (1) for the case where X and Y are both discrete, and the proof is based on (2) for the case where (X, Y) has joint PDF $f_{X,Y}$.

- Example 1. Suppose that (X, Y) has joint PDF $f_{X,Y}$, where

$$f_{X,Y}(x,y) = \begin{cases} e^{-x}e^{-y} & \text{if } x > 0 \text{ and } y > 0; \\ 0 & \text{otherwise} \end{cases}$$

  (a) Find a PDF of X.

  (b) Find $E(X)$ using (2) and compare the result with $\int_{-\infty}^{\infty} x f_X(x)dx$, where f_X is the PDF of X from Part (a).

  (c) Find $E(XY)$.

  (d) Find $E(X + XY)$.

  Ans. (a) Let $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = e^{-x}I_{(0,\infty)}(x)$ for $x \in R$. Then f_X is a PDF of X. (b) Both $\int_{R^2} x f_{X,Y}(x,y)d(x,y)$ and $\int_{-\infty}^{\infty} x f_X(x)dx$ are equal to 1. (c) 1. (d) 2.

- Joint MGF (moment generating function). Suppose that (X, Y) is a vector of random variables such that $E(e^{t_1X+t_2Y}) < \infty$ for $|t_1| < h_1$ and $|t_2| < h_2$ for some positive numbers h_1 and h_2. Let

$$M_{X,Y}(t_1,t_2) = E(e^{t_1X+t_2Y})$$

  for $(t_1, t_2) \in (-h_1, h_1) \times (-h_2, h_2)$, then $M_{X,Y}$ is called the joint MGF of (X, Y).

- Suppose that the MGF of (X, Y) is finite on $(-h_1, h_1) \times (-h_2, h_2)$ for some positive numbers h_1 and h_2. Let M_X, M_Y and $M_{X,Y}$ denote the MGF of X, Y and (X, Y) respectively. Then

$$M_X(t_1) = M_{X,Y}(t_1, 0) \text{ for } t_1 \in (-h_1, h_1)$$

and

$$M_Y(t_2) = M_{X,Y}(0, t_2) \text{ for } t_2 \in (-h_2, h_2).$$

Moreover, for nonnegative integers m, n, we have

$$E(X^m Y^n) = \frac{\partial^{m+n}}{\partial t_1^m \partial t_2^n} M_{X,Y}(t_1, t_2)\bigg|_{(t_1,t_2)=(0,0)}.$$

- Example 2. Consider the (X, Y) in Example 1.
  (a) Find the MGF of (X, Y).
  (b) Find the MGF of X.
  (c) Find $E(XY)$ and $E(X)$ using the MGF of (X, Y) and the MGF of X.

  Ans. (a) $M_{X,Y}(t_1, t_2) = 1/((1 - t_1)(1 - t_2))$ for $t_1 < 1$, $t_2 < 1$. (b) $M_X(t) = 1/(1 - t)$ for $t < 1$.

- Fact 2 Suppose that (X, Y) and (U, V) are two random vectors such that for some positive numbers h_1 and h_2, the MGF of (X, Y) and the MGF of (U, V) are equal and finite on $(-h_1, h_1) \times (-h_2, h_2)$, then (X, Y) and (U, V) have the same distribution.

- Example 3. Consider the (X, Y) in Example 1. Show that (X, Y) and (Y, X) have the same distribution.

- Example 4. Consider the (X, Y) in Example 1. Find the MGF of $X + 2Y$.

  Sol. Let $M_{X,Y}(t_1, t_2) = 1/((1 - t_1)(1 - t_2))$ for $t_1 < 1$, $t_2 < 1$, then $M_{X,Y}$ is the MGF of (X, Y) (based on the result of Example 2). For $t < 1/2$,

$$\begin{aligned} Ee^{t(X+2Y)} &= Ee^{t_1 X + t_2 Y}\big|_{(t_1,t_2)=(t,2t)} \\ &= M_{X,Y}(t, 2t) \\ &= 1/((1 - t)(1 - 2t)). \end{aligned}$$

  Let $M(t) = 1/((1 - t)(1 - 2t))$ for $t < 1/2$, then M is the MGF of $X + 2Y$.

- Notation. Suppose that $(X, Y)^T$ is a random vector (column vector). Then $E((X, Y)^T) = (E(X), E(Y))^T$.