Some special expectations

- Mean and variance. Suppose that X is a random variable. $E(X)$ is also called the mean of X and is often denoted by μ or μ_X. The variance of X, denoted by $Var(X)$, is defined by

$$Var(X) = E((X-\mu)^2),$$

where $\mu = E(X)$.

  - $Var(X)$ is often denoted by σ^2 or σ_X^2.
  - The quantity $\sqrt{Var(X)}$ is called the standard deviation of X.

- Example 1. Suppose that X has PDF f_X, where

$$f_X(x) = \begin{cases} 1 & \text{if } x \in (0,1); \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X)$, $Var(X)$ and the standard deviation of X.

Ans. $E(X) = 1/2$, $Var(X) = 1/12$ and the standard deviation of X is $1/\sqrt{12}$.

- The following result is equivalent to Theorem 1.9.1 in the text:

Fact 1 *Suppose that X is a random variable with finite mean and finite variance. Then for constants a and b,*

$$Var(aX) = a^2 Var(X)$$

and

$$Var(X+b) = Var(X).$$

- Example 2. Consider the X in Example 1. Find $E(-2X+1)$ and $Var(-2X+1)$.

Sol. $E(-2X+1) = -2E(X)+1 = -2(1/2)+1 = 0$.

$$Var(-2X+1) = Var(-2X) = (-2)^2 Var(X) = 4(1/12) = 1/3.$$

- Suppose that X is a random variable. For a positive integer k, the expectation $E(X^k)$ is called the k-th moment of X.

- Suppose that X is a random variable. The moment generating function (MGF) of X, denote by M_X, is defined as

$$M_X(t) = E(e^{tX}),$$

for $t \in (-\infty, \infty)$ such that $E(e^{tX})$ is finite.

- Suppose that X has MGF M_X and $M_X(t)$ is finite for $t \in (-h, h)$ for some $h > 0$. Then
$$M_X^{(k)}(t) = E(X^k e^{tX}) \text{ for } t \in (-h, h) \tag{1}$$
for every nonnegative integer k. From (1),
$$M_X^{(k)}(0) = E(X^k)$$
for every nonnegative integer k.

- Example 3. Suppose that n is a positive integer and p is a constant in $(0, 1)$. Suppose that X is a discrete random variable with PMF p_X, where
$$p_X(x) = C_x^n p^x (1-p)^{n-x}$$
for $x \in \{0, 1, \ldots, n\}$ and $p_X(x) = 0$ if $x \notin \{0, 1, \ldots, n\}$. Find the MGF of X, $E(X)$ and $Var(X)$.

- Example 4. Suppose that X is a random variable with PDF f_X, where
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
for $x \in (-\infty, \infty)$. Find the MGF of X, $E(X)$ and $Var(X)$.

- The following result can be also found in Theorem 1.9.2 in the text.

  Fact 2 *Suppose that X and Y are two random variables with MGFs M_X and M_Y respectively. Suppose that there exists some $h > 0$ such that $M_X(t) = M_Y(t) < \infty$ for $t \in (-h, h)$, then X and Y have the same distribution (same CDF).*

- Example 5. Consider the X in Example 1. Let $Y = 1 - X$. Find the MGF of X and show that Y and X have the same CDF.