Quantile and expectation

- Let $0 < p < 1$. The quantile of order p of the distribution of a random variable X is a number ξ_p such that

$$P(X < \xi_p) \leq p$$

and

$$P(X \leq \xi_p) \geq p.$$

- Suppose that a random variable X has CDF F_X and for $p \in (0,1)$, there exists a x_0 such that

$$F_X(x_0) = p,$$

then x_0 is a quantile of order p.

- Note that a quantile of order p of a distribution is not always unique.

Example 1. Suppose that X has a PDF f_X, where

$$f_X(x) = \begin{cases} 1/2 & \text{if } 0 \leq x < 1 \text{ or } 2 \leq x < 3; \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$P(X \leq x) = \begin{cases} 0 & \text{if } x < 0; \\ x/2 & \text{if } 0 \leq x < 1; \\ 1/2 & \text{if } 1 \leq x < 2; \\ 1/2 + (x-2)/2 & \text{if } 2 \leq x < 3; \\ 1 & \text{if } x \geq 3. \end{cases}$$

and any number between 1 and 2 can be the quantile of order 0.5 of the distribution of X.

- Suppose that a random variable X has CDF F_X and for $p \in (0,1)$, there exists a unique x_0 such that $F_X(x_0) = p$, then $F_X^{-1}(p) = x_0$ and F_X^{-1} is also called the quantile function of X.
  - The value of the quantile function at p is the quantile of order p.
  - $F_X^{-1}(0.5)$ is the median of the distribution of X.
  - $F_X^{-1}(0.75) - F_X^{-1}(0.25)$ is the interquartile range of (the distribution of) X. "Interquartile range" is often abbreviated as "IQR".

- Example 2. Suppose that X has a PDF f_X, where

$$f_X(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Find the median and the interquartile range of the distribution of X.

Sol. Let F be the CDF of X, then

$$F(t) = \int_{-\infty}^{t} f_X(x)dx = \begin{cases} 0 & \text{if } t \le -1; \\ (1+t)^2/2 & \text{if } -1 < t \le 0; \\ (1/2) + t - (t^2/2) & \text{if } 0 < t \le 1; \\ 1 & \text{if } t > 1. \end{cases}$$

To find the median, we need to solve $F(t) = 0.5$ for t. Since F is a piecewise polynomial that is strictly increasing on $(-1, 1)$, we first compute the F value at the joint point 0 to determine which piece should be used to solve $F(t) = 0.5$. Direct calculation gives $F(0) = 0.5$, so 0 is the median.

Solving $F(t) = 0.75$ gives

$$(1/2) + t - (t^2/2) = 0.75$$

and $t \in (0, 1)$, which gives $t = 1 - 1/\sqrt{2}$. Solving $F(t) = 0.25$ gives

$$(1+t)^2/2 = 0.25$$

and $t \in (-1, 0]$, which gives $t = -1 + 1/\sqrt{2}$. The interquartile range is $1 - 1/\sqrt{2} - (-1 + 1/\sqrt{2}) = 2 - \sqrt{2}$.

- The expectation of a random variable X, denoted by $E(X)$, is the "long term average" of X. Suppose that X_1, X_2, ... are independent random variables such that X_i has the same distribution of X, then

$$E(X) = \lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n}.$$

  - It is not always possible to define $E(X)$. However, $E(|X|)$ can always be defined, and $E(|X|)$ is either ∞ or a finite value. When $E(|X|) < \infty$, $E(X)$ can be defined and is a finite value.
  - For the computation of $E(X)$. We will focus on the cases where X is discrete or X has a PDF.

- Suppose that X is discrete with PMF p_X. Then

$$E(X) = \sum_x xP(X = x) = \sum_x xp_X(x),$$

where the sum is over all possible values of X. Here we require that $\sum_x |x|p_X(x)$ is finite so that the sum remains the same when the terms are re-arranged.

- Example 3. Suppose that X is a discrete random variable with PMF p_X, where

$$p_X(x) = \begin{cases} 0.2 & \text{if } x = 0; \\ 0.3 & \text{if } x = 1; \\ 0.5 & \text{if } x = 2; \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X)$.

Sol. $E(X) = 0 \times 0.2 + 1 \times 0.3 + 2 \times 0.5 = 1.3$.

- Example 4. Consider the X in Example 3. Let $Y = (X-1)^2$. Find $E(Y)$.

  Sol. The possible values of X are 0, 1, 2, so the set of all possible values of Y is $\{(x-1)^2 : x \in \{0,1,2\}\} = \{0,1\}$. Since

$$\begin{aligned} P(Y=y) &= P((X-1)^2 = y) \\ &= \begin{cases} P(X=1) = 0.3 & \text{if } y = 0; \\ P(X=0) + P(X=2) = 0.2 + 0.5 = 0.7 & \text{if } y = 1; \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

  we have $E(Y) = 0 \times P(Y=0) + 1 \times P(Y=1) = 0.7$.

- When X is a discrete random variable with PMF p_X, $E(g(X))$ can also be computed as

$$E(g(X)) = \sum_x g(x) p_X(x), \qquad (1)$$

  where the sum is over all possible values of X. Here we require that $E(|g(X)|) = \sum_x |g(x)| p_X(x)$ is finite so that the sum remains the same when the terms are re-arranged.

- Example 5. In Example 4, $E(Y) = E((X-1)^2)$ can also be obtained using

$$\begin{aligned} E((X-1)^2) &= \sum_{x=0}^{2} (x-1)^2 p_X(x) \\ &= (0-1)^2 \times P(X=0) + (1-1)^2 \times P(X=1) + (2-1)^2 \times P(X=2) \\ &= 1 \times (P(X=0) + P(X=2)) = 0.7. \end{aligned}$$

- When X is discrete with PMF p_X and $g(X)$ can take positive or negative values, one way to check whether $E(|g(X)|) < \infty$ is to compute $E(g(X)) = \sum_{x:p_X(x)>0} g(x) p_X(x)$ using

$$E(g(X)) = \underbrace{\sum_{x:p_X(x)>0, g(x)>0} g(x) p_X(x)}_{I} + \underbrace{\sum_{x:p_X(x)>0, g(x)<0} g(x) p_X(x)}_{II}.$$

  - If both I and II are finite, then $E(|g(X)|) = I - II < \infty$ and $E(g(X)) = I + II$.
  - If $I = \infty$ and II is finite, then $E(|g(X)|) = \infty$ and $E(g(X)) = \infty$.
  - If $II = -\infty$ and I is finite, then $E(|g(X)|) = \infty$ and $E(g(X)) = -\infty$.
  - If $I = \infty$ and $II = -\infty$, $E(|g(X)|) = \infty$ and $E(g(X))$ cannot be defined.

- Example 6. Suppose that X is a discrete random variable with PMF p_X, where

$$p_X(x) = \begin{cases} \frac{c}{x^2} & \text{if } x \text{ is an integer and } x \neq 0; \\ 0 & \text{otherwise,} \end{cases}$$

and $c = 1/(2\sum_{k=1}^{\infty} k^{-2})$. Find $E(X)$.

Sol.

$$\begin{aligned} E(X) &= \sum_{x:\ x \text{ is an integer and } x \neq 0} x p_X(x) \\ &= \sum_{x=1}^{\infty} x\left(\frac{c}{x^2}\right) + \sum_{x=-1}^{-\infty} x\left(\frac{c}{x^2}\right), \end{aligned}$$

where

$$\sum_{x=1}^{\infty} x\left(\frac{c}{x^2}\right) = c\sum_{k=1}^{\infty}\frac{1}{k} = \infty$$

and

$$\sum_{x=-1}^{-\infty} x\left(\frac{c}{x^2}\right) \overset{k=-x}{=} c\sum_{k=1}^{\infty}\left(-\frac{1}{k}\right) = -\infty.$$

$E(X) = \infty + (-\infty)$ cannot be defined.

- When X has PDF f_X,

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)dx.$$

Here we require that $\int_{-\infty}^{\infty} |x| f_X(x)dx < \infty$.

- Example 7. Suppose that X has PDF f_X, where

$$f_X(x) = \begin{cases} 1 & \text{if } x \in (0,1); \\ 0 & \text{otherwise.} \end{cases}$$

Find $E(X/(1+X))$.

Sol. Let $Y = X/(1+X)$, we will find $E(Y)$ by finding the PDF of Y. Let $S_X = \{x : f_X(x) > 0\}$, then $S_X = (0,1)$. Let $g(x) = x/(1+x)$ for $x \in (0,1)$, then $Y = g(X)$ and $g' > 0$ on $(0,1)$, so Y has a PDF f_Y given by

$$\begin{aligned} f_Y(y) &= \begin{cases} f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| & \text{if } y \in \{x/(1+x) : x \in (0,1)\}; \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} f_X(y/(1-y))\left|\frac{d}{dy}\left(\frac{1}{1-y} - 1\right)\right| & \text{if } y \in (0,0.5); \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1/(1-y)^2 & \text{if } y \in (0,0.5); \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$$E(Y) = \int_0^{0.5} y \left(\frac{1}{(1-y)^2} \right) dy = 1 - \ln(2).$$

- When X is a random variable with PDF f_X, $E(g(X))$ can also be computed as
$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x) dx. \tag{2}$$
Here we require that $E(|g(X)|) = \int_{-\infty}^{\infty} |g(x)| f_X(x) dx$ is finite.

- Example 8. In Example 7, $E(X/(1+X))$ can also be computed using
$$\begin{aligned} E(X/(1+X)) &= \int_{-\infty}^{\infty} \left(\frac{x}{1+x} \right) f_X(x) dx \\ &= \int_0^1 \left(\frac{x}{1+x} \right) dx \\ (y = 1+x) \quad &= \int_1^2 \frac{(y-1)}{y} dy \\ &= 1 - \ln(2). \end{aligned}$$

- When X is a continuous random variable with PDF f_X and $g(X)$ can take positive or negative values, one way to check whether $E(|g(X)|) < \infty$ is to compute $E(g(X)) = \int g(x) f_X(x) dx$ using
$$E(g(X)) = \underbrace{\int_{x:g(x)>0} g(x) f_X(x) dx}_{I} + \underbrace{\int_{x:g(x)<0} g(x) f_X(x) dx}_{II}.$$
  - If both I and II are finite, then $E(|g(X)|) = I - II < \infty$ and $E(g(X)) = I + II$.
  - If $I = \infty$ and II is finite, then $E(|g(X)|) = \infty$ and $E(g(X)) = \infty$.
  - If $II = -\infty$ and I is finite, then $E(|g(X)|) = \infty$ and $E(g(X)) = -\infty$.
  - If $I = \infty$ and $II = -\infty$, $E(|g(X)|) = \infty$ and $E(g(X))$ cannot be defined.

- Example 9. Suppose that X has PDF f_X, where
$$f_X(x) = \frac{1}{\pi(1+x^2)} \text{ for } x \in (-\infty, \infty).$$
Find $E(X)$.

  Sol.
$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_0^{\infty} x f_X(x) dx + \int_{-\infty}^{0} x f_X(x) dx, \end{aligned}$$

where

$$\int_0^\infty x f_X(x)dx = \int_0^\infty x\left(\frac{1}{\pi(1+x^2)}\right)dx = \infty$$

and

$$\int_{-\infty}^0 x f_X(x)dx = \int_{-\infty}^0 x\left(\frac{1}{\pi(1+x^2)}\right)dx = -\infty.$$

$E(X) = \infty + (-\infty)$ cannot be defined.

- Properties of expectation. Suppose that X and Y are random variables with the same sample space, and $E(|X|)$ and $E(|Y|)$ are finite. Then (i)-(iii) hold.

  (i) $E(X+Y) = E(X) + E(Y)$.

  (ii) $E(cX) = cE(X)$ for a constant c.

  (iii) $E(k) = k$ for a constant k.

  We can verify (i) for the special case where $X = h_1(Z)$ and $Y = h_2(Z)$ for some random variable Z, where Z can be a discrete random variable with PMF p_Z or a continuous random variable with PDF f_Z. Later we will be able to prove (i) for the case where (X,Y) has joint PDF or X, Y are both discrete.

  Example 10. Suppose that X is a random variable with $E(X) = 0$ and $E(X^2) = 1$. Find $E(X-2)^2$.

  Sol. $E((X-2)^2) = E(X^2 - 4X + 4) = E(X^2) - 4E(X) + 4 = 1 - 4\cdot 0 + 4 = 5$.

- $E(X)$ is finite if and only if $E|X| < \infty$. In such case, we say that the random variable X is integrable.

- Suppose that X is a random variable such that

$$P(X \in [m, M]) = 1,$$

  where m and M are constants. Then X is integrable and

$$m \le E(X) \le M.$$