Random variables (隨機變數)

- Suppose that $\mathcal{F}$ is a σ-field on a sample space Ω and P is a probability function defined on $\mathcal{F}$, then $(\Omega, \mathcal{F}, P)$ is called a probability space.

- Suppose that $(\Omega, \mathcal{F}, P)$ is called a probability space and X is a real-valued function defined on Ω. If for all $A \in \mathcal{B}(R)$,

$$\{X \in A\} = \{w \in \Omega : X(w) \in A\} \in \mathcal{F},$$

  then X is called a random variable on the probability space $(\Omega, \mathcal{F}, P)$.

- Example 1. Consider the experiment of tossing a fair coin twice. The sample space for the experiment is $\Omega = \{HH, HT, TH, TT\}$, where H and T indicate heads and tails respectively. Let

$$P(A) = \frac{1}{4} \cdot (\text{number of elements in A})$$

  for $A \in 2^{\Omega}$, then $(\Omega, \mathcal{F}, P)$ is a probability space. Let X be the number of heads obtained after tossing the coin twice, then X is a random variable on the probability space $(\Omega, \mathcal{F}, P)$.

- Distribution of a random variable. Suppose that X is a random variable on a probability space $(\Omega, \mathcal{F}, P)$. Define

$$Q(A) = P(X \in A)$$

  for $A \in \mathcal{B}(R)$. Then Q is a probability function on $(R, \mathcal{B}(R))$. Q is called the distribution of X, denoted by P_X.

- The distribution of a random variable X can be determined using its CDF (cumulative distribution function; 累積分布函數) F_X. The CDF of X F_X is defined by

$$F_X(x) = P(\{w \in \Omega : X(w) \leq x\}) = P(X \leq x)$$

  for $x \in R$.

- Properties of a CDF. Suppose that F is a CDF of a random variable. Then (i)-(iii) hold:

  (i) F is increasing ($F(a) \leq F(b)$ for $a \leq b$).

  (ii) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

  (iii) F is right continuous ($\lim_{x \to a^+} F(x) = F(a)$ for all $a \in R$).

  Note. If F is a function defined on R satisfying (i)–(iii), then F is a CDF.

- Example 2. Which of the following F is a CDF?

(a)

$$F(x) = \begin{cases} 0 & \text{if } x \le 0; \\ 1 & \text{if } x > 0. \end{cases}$$

(b)

$$F(x) = \begin{cases} 2 - |x| & \text{if } -2 \le x \le 1; \\ 1 & \text{if } x > 1; \\ 0 & \text{if } x < -2. \end{cases}$$

(c) $F(x) = x$ for $x \in (-\infty, \infty)$.

(d)

$$F(x) = \begin{cases} 0 & \text{if } x \le 0; \\ 1 - e^{-x} & \text{if } x > 0. \end{cases}$$

Ans: (d)

- Suppose that F is the CDF of a random variable X. Then for $a \in R$,

$$P(X = a) = F(a) - F(a^-), \tag{1}$$

where $F(a^-)$ denotes $\lim_{x \to a^-} F(x)$. The proof of (1) is based on the continuity of a probability function.

- Example 3. Suppose that X is a random variable with CDF F, where

$$F(x) = \begin{cases} 0 & \text{if } x < 0; \\ 1/4 & \text{if } 0 \le x < 1; \\ 3/4 & \text{if } 1 \le x < 2; \\ 1 & \text{if } x \ge 2. \end{cases} \tag{2}$$

Find $P(X = 2)$.

Ans. 1/4.

- Example 4. Find the CDF of the X in Example 1.

Ans. The CDF of X is the F given in (2).

- Probability calculation using a CDF. Suppose that F_X is the CDF of some random variable X. Then for an interval I, $P(X \in I)$ can be calculated using (1) and the fact that

$$P(X \in (a, b]) = F_X(b) - F_X(a)$$

for $a < b$ and a, $b \in R$.

- Example 5. Suppose X is a random variable with CDF F_X, where

$$F_X(x) = \begin{cases} 0 & \text{if } x \le 0; \\ 1 - e^{-x} & \text{if } x > 0. \end{cases}$$

  Find $P(-1 < X < 3)$.

  Sol. Note that $1 - e^{-x}$ is a continuous function of x, so F_X is continuous on $(0, \infty)$, which implies that $P(X = 3) = F_X(3) - F_X(3^-) = 0$. Thus

$$P(-1 < X < 3) = P(-1 < X \le 3) = F_X(3) - F_X(-1) = 1 - e^{-3} - 0 = 1 - e^{-3}.$$

- Discrete random variable (離散型隨機變數). Suppose that X is a random variable on a probability space $(\Omega, \mathcal{F}, P)$.
  - If there exists a set $S \in \mathcal{B}(R)$ such that S is countable[1] and $P(X \in S) = 1$, then X is called a discrete random variable.
  - Define a function p_X: $R \to [0, 1]$ by

$$p_X(a) = P(X = a) = P(\{w \in \Omega : X(w) = a\})$$

    for $a \in R$. p_X is called the PMF (probability mass function) of X. When X is discrete, it is clear that for $A \in \mathcal{B}(R)$,

$$P(X \in A) = \sum_{a \in A \text{ and } P(X=a)>0} P(X = a) = \sum_{a \in A \text{ and } p_X(a)>0} p_X(a). \tag{3}$$

- Example 6. Consider the X in Example 3. Let $A = (0, 0.5) \cup (0.6, 1.2) \cup (1.3, 2.1)$. Find $P(X \in A)$.

  Sol. Compute $P(X = a)$ using (1) and we have $P(X = 0) = 1/4$, $P(X = 1) = 1/2$, $P(X = 2) = 1/4$, and $P(X = a) = 0$ for $a \notin \{0, 1, 2\}$. Since $P(X \in \{0, 1, 2\}) = 1/4 + 1/2 + 1/4 = 1$, X is a discrete random variable with range $\{0, 1, 2\}$. Thus

$$\begin{aligned} P(X \in A) &= \sum_{a \in A \cap \{0,1,2\}} P(X = a) \\ &= \sum_{a \in \{1,2\}} P(X = a) \\ &= P(X = 1) + P(X = 2) = \frac{3}{4}. \end{aligned}$$

- A function p_X: $R \to [0, 1]$ is the PMF of a discrete random variable X if and only if

$$\sum_{x : p_X(x) > 0} p_X(x) = 1.$$

[1] A set S is countable if and only if S is a finite set or S can be expressed as $\{x_1, x_2, \ldots\}$ for some sequence $\{x_n\}_{n=1}^{\infty}$.

- Probability calculation using a PMF based on (3).

  Example 7. Suppose that X is a discrete random variable with possible values 0, 1, 2, ... and the PMF of X is p_X, which is given by

  $$p_X(x) = c \cdot q^x$$

  for $x = 0, 1, 2, \ldots$, where c and q are positive constants and $q \in (0, 1)$. Express c as a function of q and find $P(X > 10)$.

  Sol. Solving $\sum_{x=0}^{\infty} cq^x = 1$ gives $c = 1/(\sum_{x=0}^{\infty} q^x) = 1 - q$.

  $$P(X > 10) = \sum_{x=11}^{\infty} (1-q)q^x = \frac{(1-q)q^{11}}{1-q} = q^{11}.$$

- PMF of a random variable that is transformed from a discrete random variable. Suppose that X is a discrete random variable with PMF p_X and $Y = g(X)$, then Y is a discrete random variable with PMF p_Y, where for $y \in (-\infty, \infty)$,

  $$p_Y(y) = P(g(X) = y) = \sum_{x: p_X(x) > 0 \text{ and } g(x) = y} p_X(x).$$

- Example 8. Suppose that X is a discrete random variable with PMF p_X, where

  $$p_X(x) = \begin{cases} C_x^{10} p^x (1-p)^{10-x} & \text{if } x \in \{0, 1, \ldots, 10\}; \\ 0 & \text{otherwise.} \end{cases}$$

  Find the PMF of $Y = (X-5)^2$.

  Sol. The possible values of $Y = (X-5)^2$ are in the set

  $$\{(x-5)^2 : x \in \{0, 1, \ldots, 10\}\} = \{0^2, 1^2, \ldots 5^2\}.$$

  For $y \in \{0^2, 1^2, \ldots, 5^2\}$ and $y \neq 0$,

  $$\{x \in \{0, 1, \ldots, 10\} : (x-5)^2 = y\} = \{5 + \sqrt{y}, 5 - \sqrt{y}\},$$

  so

  $$\begin{aligned} P(Y = y) &= P((X-5)^2 = y) \\ &= \sum_{x \in \{0,1,\ldots,10\}: (x-5)^2 = y} p_X(x) \\ &= p_X(5 + \sqrt{y}) + p_X(5 - \sqrt{y}) \\ &= C_{5+\sqrt{y}}^{10} p^{5+\sqrt{y}} (1-p)^{5-\sqrt{y}} + C_{5-\sqrt{y}}^{10} p^{5-\sqrt{y}} (1-p)^{5+\sqrt{y}}. \end{aligned}$$

For $y = 0$,

$$P(Y = 0) = P((X-5)^2 = 0) = P(X = 5) = C_5^{10} p^5 (1-p)^5.$$

Let p_Y be the PMF of Y, then

$$p_Y(y) = P(Y = y) = \begin{cases} C_{5+\sqrt{y}}^{10} p^{5+\sqrt{y}} (1-p)^{5-\sqrt{y}} + C_{5-\sqrt{y}}^{10} p^{5-\sqrt{y}} (1-p)^{5+\sqrt{y}} & \text{if } y \in \{1^2, \ldots, 5^2\}; \\ C_5^{10} p^5 (1-p)^5 & \text{if } y = 0; \\ 0 & \text{otherwise.} \end{cases}$$

- Suppose that X is a random variable with CDF F_X. If F_X is continuous on $(-\infty, \infty)$, then X is called a continuous random variable. In such case, if there exists a function $f_X \geq 0$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

  for all $x \in (-\infty, \infty)$, then f_X is called a PDF (probability density function) of X.

- For a random variable X with PDF f_X, we have

$$P(X \in A) = \int_A f_X(t)dt \tag{4}$$

  for every $A \in \mathcal{B}(R)$.

- Example 9. Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = \begin{cases} 0 & \text{if } x \leq 0; \\ e^{-x} & \text{if } x > 0. \end{cases}$$

  Find $P(-1 < X < 3)$.

  Solution.

$$\begin{aligned} P(-1 < X < 3) &= \int_{-1}^{3} f_X(t)dt \\ &= \int_{-1}^{0} f_X(t)dt + \int_{0}^{3} f_X(t)dt \\ &= 0 + \int_{0}^{3} e^{-t}dt = 1 - e^{-3}. \end{aligned}$$

- PDF of a random variable that is transformed from a random variable with PDF. Suppose that X is a random variable with PDF f_X and $Y = g(X)$, where g is a one-to-one function on $S_X = \{x : f_X(x) > 0\}$ and $g' \neq 0$ on S_X. Then Y is a random variable with PDF f_Y, where

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & \text{if } y \in \{g(x) : x \in S_X\}; \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Proof for the case where S_X is an open interval (a,b), $g' > 0$ on S_X, $g(a) = \lim_{x\to a^+} g(x)$ and $g(b) = \lim_{x\to b^-} g(x)$. In this case, we have $\{g(x) : x \in S_X\}$ is the interval $(g(a), g(b))$. For $y \in (g(a), g(b))$, we have $g^{-1}(y) \in S_X$ and

$$\begin{aligned} P(Y \le y) &= P(X \le g^{-1}(y)) \\ &= \int_{-\infty}^{g^{-1}(y)} f_X(t)dt \\ &= \int_{a}^{g^{-1}(y)} f_X(t)dt \\ (s = g(t)) \quad &= \int_{g(a)}^{y} f_X(g^{-1}(s))\frac{d}{ds}g^{-1}(s)ds \\ &= \int_{-\infty}^{y} f_Y(s)ds, \end{aligned}$$

where f_Y is given in (5), that is,

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \frac{d}{dy}g^{-1}(y) & \text{if } y \in I; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $y \le g(a)$, we have

$$P(Y \le y) = 0 = \int_{-\infty}^{y} f_Y(s)ds$$

and for $y \ge g(b)$, we have

$$P(Y \le y) = 1 = \int_{-\infty}^{y} f_Y(s)ds,$$

so the f_Y in (5) is a PDF of Y.

- Example 10. Suppose that X is a random variable with PDF f_X, where

$$f_X(x) = \begin{cases} 0 & \text{if } x \le 0; \\ e^{-x} & \text{if } x > 0, \end{cases}$$

Let F be the CDF of X. Find a PDF of $Y = F(X)$.

Sol. Compute $F(t) = \int_{-\infty}^{t} f_X(x)dx$ for $t \in (-\infty, \infty)$, then we have

$$F(t) = \begin{cases} 0 & \text{if } t \le 0; \\ 1 - e^{-t} & \text{if } t > 0. \end{cases}$$

Since F is strictly increasing on $S_X = \{x : f_X(x) > 0\} = (0, \infty)$ and $F'(x) = f_X(x) > 0$ for $x > 0$, Y is a random variable with PDF f_Y, where

$$\begin{aligned} f_Y(y) &= \begin{cases} f_X(F^{-1}(y)) \left|\frac{d}{dy}F^{-1}(y)\right| & \text{if } y \in \{F(x) : x \in S_X\}; \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1 & \text{if } y \in (0,1); \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here the second equality holds since

$$\{F(x) : x \in S_X\} = \{1 - e^{-x} : x \in (0, \infty)\} = (0, 1)$$

and for $y \in (0, 1)$,

$$\frac{d}{dy}F^{-1}(y) = \frac{1}{F'(F^{-1}(y))} = \frac{1}{f_X(F^{-1}(y))}.$$

- When $Y = g(X)$ and g is not a one-to-one function, one can compute $P(Y \leq t)$ and express the probability as $\int_{-\infty}^{t} f_Y(y)dy$ for some nonnegative function f_Y, then f_Y is a PDF of Y.

Example 11. Suppose that X has PDF f_X, where

$$f_X(x) = \begin{cases} 2(1+x)/3 & \text{if } -1 < x \leq 0; \\ (2-x)/3 & \text{if } 0 < x < 2; \\ 0 & \text{if } x \notin (-1, 2), \end{cases}$$

Let $Y = X^2$. Find the PDF of Y.

Sol. Note that $f_X(x) = 0$ for $x \notin (-1, 2)$, so $P(-1 < X < 2) = 1$ and $P(0 \leq Y < 4) = 1$. Since $P(Y = 0) = P(X = 0) = 0$, we have $P(0 < Y < 4) = 1$. For $t \in (0, 4)$,

$$\begin{aligned} P(Y \leq t) &= P(-\sqrt{t} \leq X \leq \sqrt{t}) \\ &= \int_{-\sqrt{t}}^{0} f_X(x)dx + \int_{0}^{\sqrt{t}} f_X(x)dx, \end{aligned}$$

where

$$\begin{aligned} \int_{-\sqrt{t}}^{0} f_X(x)dx &= \begin{cases} \int_{-\sqrt{t}}^{0} 2(1+x)/3dx & \text{if } t \in (0,1); \\ \int_{-1}^{0} 2(1+x)/3dx & \text{if } t \in [1,\infty) \end{cases} \\ (y = x^2, x = -\sqrt{y}) &= \begin{cases} \int_0^t \frac{2(1-\sqrt{y})}{3}\frac{1}{2\sqrt{y}}dy & \text{if } t \in (0,1); \\ \int_0^1 \frac{2(1-\sqrt{y})}{3}\frac{1}{2\sqrt{y}}dy & \text{if } t \in [1,\infty) \end{cases} \\ &= \int_0^t \frac{1-\sqrt{y}}{3\sqrt{y}} I_{(0,1)}(y)dy \end{aligned}$$

and

$$\begin{aligned} \int_{0}^{\sqrt{t}} f_X(x)dx &= \begin{cases} \int_{0}^{\sqrt{t}} (2-x)/3dx & \text{if } t \in (0,4); \\ \int_{0}^{2} (2-x)/3dx & \text{if } t \in [4,\infty) \end{cases} \\ (y = x^2, x = \sqrt{y}) &= \begin{cases} \int_0^t \frac{2-\sqrt{y}}{3}\frac{1}{2\sqrt{y}}dy & \text{if } t \in (0,4); \\ \int_0^4 \frac{2-\sqrt{y}}{3}\frac{1}{2\sqrt{y}}dy & \text{if } t \in [4,\infty) \end{cases} \\ &= \int_0^t \frac{2-\sqrt{y}}{6\sqrt{y}} I_{(0,4)}(y)dy. \end{aligned}$$

Here for a set A,

$$I_A(y) = \begin{cases} 1 & \text{if } y \in A; \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$f_Y(y) = \frac{1-\sqrt{y}}{3\sqrt{y}} I_{(0,1)}(y) + \frac{2-\sqrt{y}}{6\sqrt{y}} I_{(0,4)}(y),$$

then for $t \in (0,4)$, we have

$$P(Y \le t) = \int_{-\infty}^{t} f_Y(y)dy. \tag{6}$$

Note that when $t \le 0$, (6) holds since $P(Y \le t) = 0$ and $\int_{-\infty}^{t} f_Y(y)dy = 0$. When $t \ge 4$, (6) holds since $P(Y \le t) = 1$ and $\int_{-\infty}^{t} f_Y(y)dy = 1$. Therefore, (6) holds for $t \in (-\infty, \infty)$ and f_Y is a PDF of Y.