Homework problems

Part I problems

1. Suppose that A, B are subsets of a space Ω. Let $\mathcal{F}$ be the smallest σ-fied on Ω that contains A and B. List all the sets in $\mathcal{F}$. You may write down your answer directly without justification.

2. Suppose that $\mathcal{F}$ is a σ-field on Ω and μ is a measure on $(\Omega, \mathcal{F})$. Suppose that $\{A_n\}_{n=1}^{\infty}$ is an increasing sequences of sets in $\mathcal{F}$. Show that

$$\mu\left(\cup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

3. Suppose that $(\Omega, \mathcal{G})$ and $(\Lambda, \mathcal{A})$ are measurable spaces, and $f\colon\ \Omega \to \Lambda$ is a function. Let

$$\mathcal{F} = \left\{A \in \mathcal{A} : f^{-1}(A) \in \mathcal{G}\right\}.$$

Show that $\mathcal{F}$ is a σ-field on Λ.

Remark. We can apply the result in Problem 3 with $(\Omega, \mathcal{G}) = (R^m, \mathcal{B}(R^m))$ and $(\Lambda, \mathcal{A}) = (R^k, \mathcal{B}(R^k))$ to derive the following result.

Fact 1 For a function $f\colon\ R^m \to R^k$ that is continuous on R^m, f is measurable from $(R^m, \mathcal{B}(R^m))$ to $(R^k, \mathcal{B}(R^k))$.

4. Suppose that $(\Omega, \mathcal{F})$, $(\Lambda, \mathcal{G})$, and $(\Delta, \mathcal{H})$ are measurable spaces, and f and g are two functions such that f is measurable from $(\Omega, \mathcal{F})$ to $(\Lambda, \mathcal{G})$ and g is measurable from $(\Lambda, \mathcal{G})$ to $(\Delta, \mathcal{H})$. Show that $g \circ f$ is measurable from $(\Omega, \mathcal{F})$ to $(\Delta, \mathcal{H})$.

5. Suppose that μ_1, μ_2 and ν are measures on the same measurable space, and μ_1 and μ_2 are absolutely continuous with respect to ν. Show that

$$\frac{d(\mu_1 + \mu_2)}{d\nu} = \frac{d\mu_1}{d\nu} + \frac{d\mu_2}{d\nu}.$$

That is, show that

$$\frac{d\mu_1}{d\nu} + \frac{d\mu_2}{d\nu}$$

is a density of $(\mu_1 + \mu_2)$ with respect to ν.

Part II problems

6. Suppose that $(\Omega, \mathcal{F})$ is a measurable space and f is a measurable function from $(\Omega, \mathcal{F})$ to $(R, \mathcal{B}(R))$. Suppose that $c \in R$ is a constant. Show that the function $c \cdot f$ is measurable from $(\Omega, \mathcal{F})$ to $(R, \mathcal{B}(R))$. You may make use of Fact 1 (in the remark after Problem 3) without proving it.

7. Suppose that $(\Omega, \mathcal{F})$ is a measurable space and f is a measurable function from $(\Omega, \mathcal{F})$ to $(R, \mathcal{B}(R))$. Suppose that ν is a measure on $(\Omega, \mathcal{F})$ and A is a set in $\mathcal{F}$ such that $\nu(A) = 0$.

   (a) Verify that if f is a simple function, then $\int_A f d\nu = 0$.

   (b) Use the result in Part (a) to deduce that $\int_A f d\nu = 0$.

8. Suppose that X is a discrete random variable. Let P_X be the distribution of X.

   (a) Suppose that X has m possible values $a_1, \ldots, a_m$. Let

   $$\mu = \delta_{a_1} + \cdots + \delta_{a_m}.$$

   Find a density of P_X with respect to μ. Justify your answer.

   (b) Suppose that X has possible values $a_1, a_2, \ldots$. Define

   $$\mu(A) = \sum_{i=1}^{\infty} \delta_{a_i}(A)$$

   for $A \in \mathcal{B}(R)$, then it can be shown that μ is a measure on $(R, \mathcal{B}(R))$. Find a density of P_X with respect to μ. Justify your answer.

9. Suppose that $Z \sim N(0, 1)$ and

$$X = \begin{cases} Z & \text{if } 0 < Z < 1; \\ 0 & \text{if } Z \leq 0; \\ 1 & \text{if } Z \geq 1. \end{cases}$$

Find a density of X (of the distribution of X) with respect to $\lambda + \delta_0 + \delta_1$, where λ is the Lebesgue measure on $(R, \mathcal{B}(R))$.

10. Suppose that X is a random variable and the distribution of X has a density f_X with respect to λ, where λ is the Lebesgue measure on $(R, \mathcal{B}(R))$. Let $Y = X^2$.

    (a) Suppose that $f_X(x) = 0$ for $x \geq 0$. Let

    $$g(y) = \frac{f_X(-\sqrt{y})}{2\sqrt{y}} I_{(0,\infty)}(y)$$

    for $y \in R$. Show that g is a density of Y with respect to λ.

    (b) Write down a density of Y with respect to λ in general. Justify your answer.

11. Suppose that X is a random varialbe on a probability space $(\Omega, \mathcal{F}, P)$ and $E|X| < \infty$. Suppose that $\mathcal{A}$ is a sub-σ-field of $\mathcal{F}$. Show that $E(kX|\mathcal{A}) = kE(X|\mathcal{A})$ for a constant k.

Part III problem

11. Do Exercise 1 at the end of the handout "Density estimation based on basis function approximation". Provide a proper initial value for the parameter a for `optim`. Turn in your R codes in a text file so that the codes can be copied and pasted to be excuted in an R environment. The handout is available at

    https://stat.walkup.tw/teaching/math_stat_adv/handouts/sp_den.pdf