Final homework problems

- Full points = 10 points.

1. Suppose that $\{X_n\}_{n=1}^{\infty}$, $\{Y_n\}_{n=1}^{\infty}$, $\{Z_n\}_{n=1}^{\infty}$ are sequences of random variables on the same probability space. Suppose that $X_n = O_p(1)$, $Y_n = o_p(1)$ and $Z_n = O_p(1)$.

   (a) (5 pts) Show that $Y_n = O_p(1)$.

   (b) (5 pts) Show that $X_n Y_n = o_p(1)$.

   (c) (5 pts) Show that $X_n + Z_n = O_p(1)$.

   (d) (5 pts) Show that $X_n Z_n = O_p(1)$.

2. (5 pts) Suppose that $\{c_k\}_{k=1}^{\infty}$ is a sequence of positive numbers and $\{\varepsilon_k\}_{k=1}^{\infty}$ is a sequence of IID random variables such that $E(\varepsilon_1) = 0$ and $Var(\varepsilon_1) = 1$. Let $S_n = \sum_{k=1}^{n} c_k \varepsilon_k$ for every positive integer n. Show that if

$$\lim_{n\to\infty} \frac{\max_{1\le k\le n} c_k}{\sqrt{\sum_{k=1}^{n} c_k^2}} = 0,$$

   then $S_n/\sqrt{Var(S_n)}$ converges to $N(0,1)$ in distribution as $n \to \infty$.

3. (5 pts) Suppose that $X \sim Bin(n, \theta)$, where n is a positive integer and $\theta \in (0,1)$. Show that for $k \in \{0, 1, \ldots, n\}$, $P(X \ge k)$ is an increasing function of θ.

4. (5 pts) Suppose that ϕ_1 and ϕ_2 are two tests for testing

$$H_0 : \theta \in \Theta_0 \text{ v.s. } H_1 : \theta \in \Theta_1,$$

   where $\Theta_0 \cap \Theta_1 = \emptyset$. Let β_{ϕ_1} and β_{ϕ_2} be the power functions of ϕ_1 and ϕ_2, respectively. Suppose that

$$\beta_{\phi_1}(\theta) \le \beta_{\phi_2}(\theta) \text{ for } \theta \in \Theta_0$$

   and

$$\beta_{\phi_1}(\theta) \ge \beta_{\phi_2}(\theta) \text{ for } \theta \in \Theta_1.$$

   Show that ϕ_1 is at least as good as ϕ_2 under the 0-1-1 loss (the loss is 1 if Type I error or Type II error is made, and the loss is 0 otherwise).

5. Suppose that we have a sample of IID observations $X_1, \ldots, X_n$, where

$$P(X_1 = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

for $k = 0, 1, 2, \ldots$ and $\lambda > 0$ is an unknown parameter. Consider the problem of estimating λ under a loss function L. The function L is defined so that the loss for estimating λ using a when $\lambda = \lambda_0$ is

$$L(a, \lambda_0) = (a - \lambda_0)^2/\lambda_0$$

for $a > 0$ and $\lambda_0 > 0$.

(a) (5 pts) Find the Bayes estimator for λ when the prior distribution for λ is the gamma distribution $\Gamma(\alpha, \beta)$, where $\alpha > 0$, $\beta > 0$. The Lebesgue density for distribution $\Gamma(\alpha, \beta)$ is the function $f_{\alpha,\beta}$ defined by

$$f_{\alpha,\beta}(s) = \begin{cases} s^{\alpha-1}e^{-s/\beta}/(\Gamma(\alpha)\beta^\alpha) & \text{if } s > 0; \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx.$$

(b) (5 pts) Find a minimax estimator for λ.

6. In the proof of Theorem 4.17 in the text, for $\gamma \in \partial B_n(c)$, we have

$$\begin{aligned} &\log \ell(\gamma) - \log \ell(\theta) \\ &= c\lambda^T[I_n(\theta)]^{-1/2}s_n(\theta) \\ &\quad + \frac{c^2}{2}\lambda^T[I_n(\theta)]^{-1/2}(\nabla s_n(\gamma^*)[I_n(\theta)]^{-1/2}\lambda, \end{aligned} \tag{1}$$

where γ^* is between γ and θ, $\|\lambda\| = 1$,

$$\sup_{\gamma \in \partial B_n(c)} \frac{\|\nabla s_n(\gamma^*) - \nabla s_n(\theta)\|}{n} \leq \max_{\gamma \in B_n(c)} \frac{\|\nabla s_n(\gamma) - \nabla s_n(\theta)\|}{n} \tag{2}$$

and

$$\lim_{n\to\infty} E\left(\max_{\gamma \in B_n(c)} \frac{\|\nabla s_n(\gamma) - \nabla s_n(\theta)\|}{n}\right) = 0. \tag{3}$$

(a) (5 pts) Deduce that

$$\sup_{\gamma \in \partial B_n(c)} [\log \ell(\gamma) - \log \ell(\theta)] \leq c\|[I_n(\theta)]^{-1/2} s_n(\theta)\| - [1 + o_p(1)]c^2/2$$

from (1), (2) and (3).

(b) (5 pts) Show that

$$E\|[I_n(\theta)]^{-1/2} s_n(\theta)\|^2 = k.$$

(c) (5 pts) Show that for $\varepsilon > 0$, there exists $n_0 > 1$ such that

$$P\left(\sup_{\gamma \in \partial B_n(c)} [\log \ell(\gamma) - \log \ell(\theta)] \right) \geq 1 - \varepsilon$$

using the results in Part (a) and Part (b).