

Final homework problems

- Notation.

- For $\mathcal{C}$: a collection of sets in a space Ω , $\sigma(\mathcal{C})$ denotes the σ -field generated by $\mathcal{C}$ (the smallest σ -field containing $\mathcal{C}$).
- $\mathcal{B}(R)$ denotes the Borel σ -field on R (the σ -field generated by open sets in R).

1. Suppose that $\Omega = \{1, 2, 3, 4\}$ and Y is a function on Ω defined by $Y(1) = 4$, $Y(2) = 5$ and $Y(3) = Y(4) = 6$. Let $\sigma(Y) = \{Y^{-1}(B) : B \in \mathcal{B}(R)\}$ be the smallest σ -field $\mathcal{F}$ such that Y is measurable from $(\Omega, \mathcal{F})$ to $(R, \mathcal{B}(R))$.

(a) List the events in $\sigma(Y)$. Justify your answer.

(b) Suppose that X is measurable from $(\Omega, \sigma(Y))$ to $(R, \mathcal{B}(R))$. Explain why $X = h(Y)$ for some real-valued function h defined on $\{4, 5, 6\}$.

2. Let $\mathcal{C} = \{(-\infty, t] : t \in R\}$. Show that $\sigma(\mathcal{C}) = \mathcal{B}(R)$.
3. Let $\mathcal{C} = \{(-\infty, x] \times (-\infty, y] : x, y \in R\}$. Show that for finite intervals (a, b) and (c, d) , we have

$$(a, b) \times (c, d) \in \sigma(\mathcal{C}).$$

4. Suppose that f is a function defined on a space Ω to another space Λ , and B , B_1 , B_2 , $\dots$, are subsets of Λ . Show that

$$f^{-1}(B^c) = (f^{-1}(B))^c$$

and

$$f^{-1}(\cup_{i=1}^{\infty} B_i) = \cup_{i=1}^{\infty} f^{-1}(B_i).$$

5. Let $\bar{R} = R \cup \{-\infty, \infty\}$, $A_1 = \{-\infty\}$, $A_2 = \{\infty\}$, and $\mathcal{C} = \mathcal{B}(R) \cup \{A_1, A_2\}$. Let

$$\bar{\mathcal{B}} = \{B \cup D : B \in \mathcal{B}(R) \text{ and } D \in \{A_1, A_2, A_1 \cup A_2, \emptyset\}\}.$$

Show that $\bar{\mathcal{B}}$ is a σ -field and $\sigma(\mathcal{C}) = \bar{\mathcal{B}}$.

6. Suppose that f is a real-valued continuous function on R , X is measurable from a measurable space $(\Omega, \mathcal{F})$ to $(\bar{R}, \bar{\mathcal{B}})$, $A \in \mathcal{F}$, and $c_1, c_2 \in \bar{\mathcal{B}}$. Define

$$Y(\omega) = \begin{cases} h(X(\omega)) & \text{if } X(\omega) \in R; \\ c_1 & \text{if } X(\omega) = -\infty; \\ c_2 & \text{if } X(\omega) = \infty; \end{cases}$$

Show that Y is measurable from $(\Omega, \mathcal{F})$ to $(\bar{R}, \bar{\mathcal{B}})$.

7. Suppose that f is measurable from a measurable space $(\Omega, \mathcal{F})$ to $(\bar{R}, \bar{\mathcal{B}})$, where $\bar{R}$ and $\bar{\mathcal{B}}$ are defined in Problem 5. Let

$$f^-(\omega) = \begin{cases} -f(\omega) & \text{if } f(\omega) < 0; \\ 0 & \text{if } f(\omega) \geq 0. \end{cases}$$

Show that f^- is measurable from $(\Omega, \mathcal{F})$ to $(\bar{R}, \bar{\mathcal{B}})$.

8. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are collections of some subsets of Ω_1 and Ω_2 , respectively. Show that $\sigma(\sigma(\mathcal{C}_1) \times \sigma(\mathcal{C}_2)) = \sigma(\mathcal{C}_1 \times \mathcal{C}_2)$.