Homework problems

- Note. You only need to turn in two of the following five problems. Each problem is worth 8 points.

1. Verify the following results in (a) and (b). For (a), you may verify the result numerically and provide your codes or provide a proof.

   (a) Let $i = \sqrt{-1}$. Then

   $$\left| e^{iy} - \left(1 + iy - \frac{y^2}{2}\right) \right| \le \min\left(y^2, |y|^3\right)$$

   for $y \in R$.

   (b) Suppose that $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ are complex numbers such that $|a_j| = 1 = |b_j|$ for $j = 1, \ldots, m$. Then

   $$\left| \prod_{j=1}^{m} a_j - \prod_{j=1}^{m} b_j \right| \le \sum_{j=1}^{m} |a_j - b_j|.$$

2. Suppose that $X_1, \ldots, X_n$ are IID observations and the distribution of X_1 is the uniform distribution on $[0, \theta]$, where $\theta > 0$. Consider the problem of estimating θ based on the data $X_1, \ldots, X_n$. Let $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $\hat{\theta} = \max_{1 \le i \le n} X_n$.

   (a) Show that both $2\bar{X}$ and $\hat{\theta}$ are consistent estimators of θ.

   (b) Find an asymptotic relative efficiency of $2\bar{X}$ (as an estimator of θ) relative to $\hat{\theta}$. Determine whether $\hat{\theta}$ is asymtotically more efficient than $2\bar{X}$.

3. Suppose that $X_1, \ldots, X_n$ are IID observations and the distribution of X_1 is the uniform distribution on $(-\theta, \theta)$, where $\theta > 0$. For $c \in R$, define the transform g_c by

   $$g_c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n)$$

   for $(x_1, \ldots, x_n) \in R^n$. Let $\mathcal{G} = \{g_c : c > 0\}$, then it can be shown that $\mathcal{G}$ is a group. Consider the problem of estimating θ based on $X_1, \ldots, X_n$.

   (a) Suppose the loss function for the estimation problem is the squared error loss. Explain why the problem is not invariant under $\mathcal{G}$.

(b) Propose a loss function for the estimation problem so that the problem is invariant under $\mathcal{G}$. Justify your answer.

4. Suppose that $X_1, \ldots, X_n$ are IID observations and the distribution of X_1 is the binomial distribution of size m and success prbability θ, where m is known and $\theta \in (0,1)$. That is, X_1 takes values in $\{0,1,\ldots,m\}$ with probability one and for $k \in \{0,1,\ldots,m\}$,

$$P(X_1 = k) = \frac{m!}{k!(m-k)!}\theta^k(1-\theta)^{m-k}.$$

Find a minimax estimator of θ based on the data $X_1, \ldots, X_n$ under squared error loss.

5. Suppose that $X_1, \ldots, X_n$ are IID, $E(X_1) = 0$ and $Var(X_1) = 1$. Suppose that $\{c_i\}_{i=1}^{\infty}$ is a sequence of constants. Let $Y_n = \sum_{i=1}^{n} c_i X_i$. Give condition(s) on c_i's such that

$$\frac{Y_n}{\sqrt{Var(Y_n)}} \xrightarrow{\mathcal{D}} N(0,1)$$

as $n \to \infty$. Try to simplify your condition(s) as much as possible.