Homework problems

- Note. You only need to turn in two of the following five problems. Each problem is worth 8 points.
- Notation.
  - For $c \in R$, δ_c is the measure on $(R, \mathcal{B}(R))$ defined by

$$\delta_c(A) = \begin{cases} 1 & \text{if } c \in A; \\ 0 & \text{if } c \notin A. \end{cases} \tag{1}$$

    for $A \in \mathcal{B}(R)$.

1. Suppose that f_X is a nonnegative function measurable from $(R, \mathcal{B}(R))$ to $(R, \mathcal{B}(R))$. Let λ denote the Lebesgue measure on $\mathcal{B}(R)$. Show that for $t > 0$,

$$\int_{(0,t)} \frac{f_X(-\sqrt{y})}{2\sqrt{y}} d\lambda(y) = \int_{(-\sqrt{t},0)} f_X d\lambda.$$

2. Suppose that $(X_{i,1}, X_{i,2}, Y_i)$: $i = 1, \ldots, n$ are IID observations for (X_1, X_2, Y), where

$$Y = a_1 X_1 + a_2 X_2 + \varepsilon,$$

   X_1, X_2 and ε are independent random variables, $\varepsilon \sim N(0, \sigma^2)$, and X_1 and X_2 have Lebesgue PDFs f_1 and f_2 respectively. Suppose that $\sigma > 0$ is known and we would like to estimate the parameters a_1 and a_2 using a Bayesian approach based on the data $\{(X_{i,1}, X_{i,2}, Y_i)\}_{i=1}^n$. Let

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \text{ for } x \in R \tag{2}$$

   and define

$$f_0(a) = c_0 I_{\{0\}}(a) + (1 - c_0) I_{\{0\}^c}(a) \phi(a)$$

   for $a \in R$, where $c_0 \in (0, 1)$ is a known constant. Let λ denote the Lebesgue measure on $(R, \mathcal{B}(R))$, and define δ_0 by (1) with $c = 0$. Let $\mu_0 = \delta_0 + \lambda$ and define

$$g_0(s, t) = f_0(s) f_0(t)$$

   for $(s, t) \in R^2$. Let Π be the distribution such that

$$\frac{d\Pi}{d\mu_0} = g_0$$

   and use Π as the prior distribution of (a_1, a_2). Find the posterior density of (a_1, a_2) with respect to $\mu_0 \times \mu_0$.

3. Suppose that U, Z_0 and Z_1 are independent random variables on the probability space $(\Omega, \mathcal{F}, P)$, and ν is a σ-finite measure on $(R, \mathcal{B}(R))$. Suppose that $P(U = 1) = 1 - P(U = 0) \in (0, 1)$ and for $i \in \{0, 1\}$, Z_i has a PDF f_i with respect to ν. Define

$$X = \begin{cases} Z_0 & \text{if } U = 0; \\ Z_1 & \text{if } U = 1. \end{cases}$$

Let $\pi_0 = P(U = 0)$, $\pi_1 = P(U = 1)$ and define

$$f(x) = \pi_0 f_0(x) + \pi_1 f_1(x)$$

for $x \in R$. Show that f is a PDF of X with respect to ν.

4. Suppose that U and Z are independent random variables on the probability space $(\Omega, \mathcal{F}, P)$, $P(U = 1) = 1 - P(U = 0) \in (0, 1)$ and $Z \sim N(0, \sigma^2)$. Define

$$X = \begin{cases} 0 & \text{if } U = 0; \\ Z & \text{if } U = 1. \end{cases}$$

Let $\pi_0 = P(U = 0)$, $\pi_1 = P(U = 1)$ and define

$$f(x) = \pi_0 I_{\{0\}}(x) + \pi_1 I_{\{0\}^c}(x)\phi(x)$$

for $x \in R$, where ϕ is defined in (2).

(a) For $t \in R$, let

$$F(t) = \int_{(-\infty,t]} f(x) d\mu_0(x)$$

where μ_0 is defined in Problem 2. Find $F(t)$ for $t \in R$.

(b) Verify that

$$P(X \le t) = F(t) \text{ for } t \in \{0.5, 1, 1.5\}. \tag{3}$$

You may verify (3) by applying the result in Problem 3 or finding $P(X \le t)$ via direct calculation or numerical approximation. For numerical approximation of $P(X \le t)$, you need to first propose an estimator of $P(X \le t)$ based on IID data from the distribution of X, and then generate IID data from the distribution of X to obtain the estimated $P(X \le t)$ based on the generated data. Then, $P(X \le t)$ can be approximated by its estimated value.

5. Suppose that we have n IID observations $X_1, \ldots, X_n$, where X_1 has a Lebesgue PDF f. Suppose that $f(x) = 0$ for $x \notin (0,1)$ and f is positive and continuous on $(0,1)$. To construct an estimator of f, let

$$k(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1; \\ 0 & \text{if } |x| > 1. \end{cases}$$

and let $\{h_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that

$$\lim_{n \to \infty} h_n = 0$$

and

$$\lim_{n \to \infty} nh_n = \infty.$$

For $x_0 \in (0,1)$, let

$$\hat{f}(x_0) = \frac{1}{nh_n} \sum_{i=1}^{n} k\left(\frac{x_0 - X_i}{h_n}\right).$$

(a) Show that for $x_0 \in (0,1)$,

$$E\left[k^j\left(\frac{x_0 - X_i}{h_n}\right)\right] = f(x_0)h_n \int_{(x_0-1)/h_n}^{x_0/h_n} k^j(u)du + o(h_n)$$

for $j = 1, 2$.

(b) For $x_0 \in (0,1)$, show that $\hat{f}(x_0)$ converges to $f(x_0)$ in probability as $n \to \infty$.

(c) Suppose that $x_0 \in (0,1)$. Show that

$$\frac{\hat{f}(x_0) - E[\hat{f}(x_0)]}{Var\left(\hat{f}(x_0)\right)}$$

converges to a $N(0,1)$ random variable in distribution as $n \to \infty$.

(d) Suppose that x_1 and x_2 are two points in $(0,1)$. Show that

$$\sqrt{nh_n}\begin{pmatrix} \hat{f}(x_1) - E[\hat{f}(x_1)] \\ \hat{f}(x_2) - E[\hat{f}(x_2)] \end{pmatrix}$$

converges to a bivariate normal random variable in distribution as $n \to \infty$.