Homework problems to be turned in

- Note. This problem file may be updated during the semester till Jan. 5 2022; more problems may be added.

1. Suppose that μ_1 and μ_2 are measures on a measurable space $(\Omega, \mathcal{F})$. Suppose that f is a nonnegative measurable function from $(\Omega, \mathcal{F})$ to $(R, \mathcal{B}(R))$. Show that
$$\int f d(\mu_1 + \mu_2) = \int f d\mu_1 + \int f d\mu_2.$$
Note that $\mu_1 + \mu_2$ is the measure μ such that $\mu(A) = \mu_1(A) + \mu_2(A)$ for $A \in \mathcal{F}$.

2. Suppose that U, Z_1, and Z_2 are independent random variables on the probability space $(\Omega, \mathcal{F}, P)$. Suppose that $P(U = 1) = 1 - \pi_0$ and $P(U = 2) = \pi_0$, $Z_1 \sim N(0, \sigma^2)$, and $P(Z_2 = 0) = 1$, where $\pi_0 \in (0, 1)$ and $\sigma > 0$ are constants. Define
$$X = \begin{cases} Z_1 & \text{if } U = 1; \\ Z_2 & \text{if } U = 2. \end{cases}$$
Find the density of X with respect to $\lambda + \delta_0$, where λ is the Lebesgue measure on $(R, \mathcal{B}(R))$ and $\delta_0(A) = I_A(0)$ for $A \in \mathcal{B}(R)$. You need to justify your answer by verifying that the integral of the density over A with respect to $\lambda + \delta_0$ is $P \circ X^{-1}(A)$ for $A \in \mathcal{B}(R)$.

3. Suppose that X is a random variable on the probability space $(\Omega, \mathcal{F}, P)$ and $E|X| < \infty$. Suppose that $\mathcal{A}$ is a sub-σ-field of $\mathcal{F}$. Show that for a constant a,
$$E(aX|\mathcal{A}) = aE(X|\mathcal{A}).$$

4. Suppose that (X, Y) is a random vector on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that X takes values in
$$S_X = \{x_1, x_2, \ldots\} \subset (-\infty, \infty)$$
and Y takes values in
$$S_Y = \{y_1, y_2, \ldots\} \subset R^k.$$
Suppose that $E|X| < \infty$ and $P(Y = y) > 0$ for all $y \in S_Y$. Define
$$h(y) = \sum_{x \in S_X} xP(X = x|Y = y)$$
for $y \in S_Y$. Show that $E(X|Y) = h(Y)$.

5. Suppose that X and Y are random variables. Suppose that a and b are two constants and let

$$p(x) = \frac{e^{a+bx}}{1+e^{a+bx}}$$

for $x \in (-\infty, \infty)$. Let

$$f_{Y|X=x}(y) = I_{\{1\}}(y)p(x) + I_{\{0\}}(y)(1-p(x))$$

for $x, y \in (-\infty, \infty)$. Suppose that $f_{Y|X=x}$ is a version of the condtional PDF of Y given $X = x$ with respect to $(\delta_0 + \delta_1)$ for $x \in (-\infty, \infty)$, where for $c \in (-\infty, \infty)$, δ_c denotes the measure defined by

$$\delta_c(A) = \begin{cases} 1 & \text{if } c \in A; \\ 0 & \text{if } c \notin A \end{cases}$$

for $A \in \mathcal{B}(R)$. Let λ denote the Lebesgue measure on $(R, \mathcal{B}(R))$. Suppose that the distribution of X has a PDF f_X with respect to λ. Find a version of the conditional PDF of X given $Y = y$ with respect to λ for $y \in \{0, 1\}$.

6. Suppose that $X_n \sim Bin(n, 0.5)$ for n = 1,2, ..., where $Bin(n, 0.5)$ is the binomial distribution of size n and success probability p, so

$$P(X_n = x) = C_x^n p^x (1-p)^{n-x}$$

for $x \in \{0, 1, \ldots, n\}$. Let $Z_n = (X_n - E(X_n))/\sqrt{Var(X_n)}$ and let F_n be the CDF of Z_n and let F be the CDF of the standard normal distribution. For $w \in (0, 1)$, let

$$Y_n(w) = \inf\{x : w \leq F_n(x)\}$$

and

$$Y(w) = \inf\{x : w \leq F(x)\}.$$

Write R codes to plot the graphs of Y_n and Y on $(0, 1)$ for $n = 50$ and 5000. Please turn in the R codes in a text file. The Y_n and Y are given in the proof of Theorem 25.6 (Skorohod's theorem) in the book "Probability and Measure" by Billingsley (3rd Edition) (see the file "skorohod.JPG" attached in WM5).

7. Suppose that X and Z are random vectors that take values in R^k with PDFs f_X and f_Z respectively with respect to a σ-finite measure ν. Suppose that there exists h: a measurable function from $(R^k, \mathcal{B}(R^k))$ to $(R, \mathcal{B}(R))$ such that $h > 0$ ν-a.e. and

$$f_X = h f_Z.$$

Show that h is a PDF of X with respect to P_Z: the distribution of Z.

Hint: apply the following result.

Fact 1 *Suppose that μ and ν are two measures on $(\Omega, \mathcal{F})$, $\mu \ll \nu$ and ν is σ-finite. Suppose that $h \geq 0$ is measurable from $(\Omega, \mathcal{F})$ to $(R, \mathcal{B}(R))$, then*

$$\int h \frac{d\mu}{d\nu} d\nu = \int h d\mu.$$

8. Suppose that $\{X_n\}_{n=1}^{\infty}$, $\{Y_n\}_{n=1}^{\infty}$, $\{Z_n\}_{n=1}^{\infty}$ are sequences of random variables on the same probability space. Suppose that $X_n = O_p(1)$, $Y_n = o_p(1)$ and $Z_n = o_p(1)$. Show that $Z_n = O_p(1)$ and $X_n + Y_n = O_p(1)$.

9. Suppose that $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent random variables on the same probability space and the distribution of X_i is the uniform distribution on $(-1 - (1/i), 1 + (1/i))$. Let $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. Show that

$$\frac{S_n}{\sqrt{Var(S_n)}}$$

converges to $N(0, 1)$ in distribution as $n \to \infty$.