Proof of a special case of MCT (monotone convergence theorem)

- Definition 2. Suppose that $(\Omega, \mathcal{F}, \nu)$ is a measure space and $f \geq 0$ is a Borel function defined on $\Omega$. The integral of $f$ with respect to $\nu$, denoted by $\int f d\nu$, is defined as

$$\int f d\nu = \lim_{n\to\infty} f_n,$$

where $\{f_n\}_{n=1}^{\infty}$ is an increasing sequence of nonnegative simple functions such that $\lim_{n\to\infty} f_n(w) = f(w)$ for every $w \in \Omega$.

- To see that the integral computed using Definition 2 is the same as that computed using Definition 1.4(b) in the text, we will prove a special case of the monotone convergence theorem (Theorem 1.1 (iii))

**Theorem 1** *Suppose that $\{f_n\}_{n=1}^{\infty}$ is an increasing sequence of nonnegative simple functions such that $\lim_{n\to\infty} f_n(w) = f(w) < \infty$ for every $w \in \Omega$. Then*

$$\int f d\nu = \lim_{n\to\infty} \int f_n d\nu,$$

*where the integrals are defined using Definition 1.4(b) in the text.*

The proof of Theorem 1 is adapted from the proof of Theorem 1.1 and the following results will be used.

**Fact 1** *Suppose that $f$ and $g$ are two Borel functions such that $0 \leq f \leq g$, then*

$$0 \leq \int f d\nu \leq \int g d\nu.$$

Fact 1 follows directly from the definition of an integral in Definition 1.4(b).

**Fact 2** *Suppose that $f$ and $g$ are two nonnegative simple functions, then*

$$\int (f+g) d\nu = \int f d\nu + \int g d\nu.$$

Fact 2 follows directly from the definition of the integral of a nonnegative simple function.

- Proof of Theorem 1. Suppose that $\{f_n\}_{n=1}^{\infty}$ is an increasing sequence of nonnegative simple functions such that $\lim_{n\to\infty} f_n(w) = f(w)$ for every $w \in \Omega$. We will show that

$$\lim_{n\to\infty} \int f_n d\nu \leq \int f d\nu \tag{1}$$

and

$$\lim_{n\to\infty} \int f_n d\nu \geq \int f d\nu \tag{2}$$

to complete the proof. Note that from Fact 1, $\{\int f_n d\nu\}_{n=1}^{\infty}$ is an increasing sequence in $[0,\infty) \cup \{\infty\}$ such that

$$\int f_n d\nu \leq \int f d\nu,$$

so (1) holds.

To prove (2), let $\phi$ be a simple function such that $0 \leq \phi \leq f$. We will show that

$$\int \phi d\nu \leq \lim_{n\to\infty} \int f_n d\nu, \tag{3}$$

then (2) holds since $\phi$ is arbitrary. To prove (3), let $A_\phi = \{w \in \Omega : \phi(w) > 0\}$. If $A_\phi = \emptyset$, then the simple function $\phi = 0$ and (3) holds. Below we will continue to prove (3) assuming $A_\phi \neq \emptyset$.

(i) Suppose that $\nu(A_\phi) = \infty$, let $a = \min\{\phi(w) : w \in A_\phi\}$, then $a > 0$. By Fact 1,

$$\int \phi d\nu \geq \int a I_{A_\phi} d\nu = a\nu(A_\phi) = \infty,$$

and it follows from $A_\phi \subset \cup_{n=1}^{\infty}\{f_n > a/2\}$ that

$$\int f_n I_{\{f_n > a/2\}} d\nu \geq \frac{a}{2}\nu\left(\{f_n > a/2\}\right) \to \frac{a}{2}\nu\left(\cup_n \{f_n > a/2\}\right) \geq \frac{a}{2}\nu(A_\phi) = \infty,$$

so (3) holds.

(ii) Suppose that $\nu(A_\phi) < \infty$, then by Egoroff's theorem (Egorov's theorem), for every $\varepsilon > 0$, there exists $B \subset A_\phi$ such that $\nu(B) < \varepsilon$ and $f_n$ converges to $f$ on $A_\phi \cap B^c$ uniformly, so for $\delta > 0$, there exists $N$ such that

$$|f_n(w) - f(w)| < \delta \text{ for } w \in A_\phi \cap B^c \text{ and } n > N,$$

which implies that

$$fI_{A_\phi \cap B^c} \leq (f_n + \delta) I_{A_\phi \cap B^c} \text{ for } n > N.$$

Taking the integral of the functions at both sides of the above inequality with respect to $\nu$ and apply Facts 1 and 2, we have for $n > N$,

$$\begin{aligned} \int f I_{A_\phi \cap B^c} d\nu &\leq \int (f_n + \delta) I_{A_\phi \cap B^c} d\nu \\ &= \int f_n I_{A_\phi \cap B^c} d\nu + \int \delta I_{A_\phi \cap B^c} d\nu \\ &\leq \int f_n d\nu + \delta \nu(A_\phi \cap B^c), \end{aligned}$$

which gives

$$\int \phi I_{A_\phi \cap B^c} d\nu \leq \int f I_{A_\phi \cap B^c} d\nu \leq \left( \lim_{n\to\infty} \int f_n d\nu \right) + \delta \nu(A_\phi \cap B^c).$$

Since $\delta > 0$ is arbitrary, we have

$$\int \phi I_{A_\phi \cap B^c} d\nu \leq \lim_{n\to\infty} \int f_n d\nu. \tag{4}$$

Moreover, since $\int \phi d\nu = \int \phi I_{A_\phi} d\nu + \int \phi I_{A_\phi{}^c} d\nu$ and $\int \phi I_{A_\phi{}^c} d\nu = 0$, we have

$$\begin{aligned} \int \phi d\nu &= \int \phi I_{A_\phi} d\nu \\ &= \int \phi I_B d\nu + \int \phi I_{A_\phi \cap B^c} d\nu. \end{aligned} \tag{5}$$

Note that

$$\int \phi I_B d\nu \leq \max\{w \in B^c : \phi(w)\} \nu(B),$$

so (5) gives

$$\begin{aligned} \int \phi d\nu &\leq \max\{w \in B^c : \phi(w)\}\varepsilon + \int \phi I_{A_\phi \cap B^c} d\nu \\ &\overset{(4)}{\leq} \max\{w \in B^c : \phi(w)\}\varepsilon + \lim_{n\to\infty} \int f_n d\nu. \end{aligned}$$

Since $\varepsilon > 0$ can be made arbitrarily small, (3) holds.

From the above discussion, (3) holds for Cases (i) and (ii), so we have prove (3), which implies (2), so the proof of Theorem 1 is complete since both (1) and (2) hold.

- The result in Theorem 1 can be extended to the case where $f(w)$ is allowed to be $\infty$ for some $w \in \Omega$. To prove the extension version, the following results are used:

  **Fact 3** *Suppose that $f$ and $g$ are nonnegative Borel functions such that $\nu(\{w : g(w) \neq f(w)\}) = 0$, then*

  $$\int f d\nu = \int g d\nu.$$

  To prove Fact 3, it is easier to prove Fact 3 for the special case where $f$ and $g$ are nonnegative simple functions, and then apply it to prove Fact 3 for the general case.

  **Fact 4** *Suppose that $\lim_{n\to\infty} f_n(w) = \infty$ for $w \in A$ and $\nu(A) < \infty$. Then for $\varepsilon > 0$, there exists $B \subset A$ such that $\nu(B) < \varepsilon$ and for $M > 0$, there exists $N$ such that for $n > N$,*

  $$f_n(w) > M \text{ for } w \in A \cap B^c.$$

  The proof of Fact 4 is left as an exercise.

- Proof of Theorem 1 for the case where $f(w)$ is allowed to be $\infty$ for some $w \in \Omega$. Let $A = \{w \in \Omega : f(w) = \infty\}$. Suppose that $\{f_n\}_{n=1}^{\infty}$ is an increasing sequence of nonnegative simple functions such that $\lim_{n\to\infty} f_n(w) = f(w)$ for every $w \in \Omega$.

  (i) Suppose that $\nu(A) = 0$. Then $\{f_n I_{A^c}\}_{n=1}^{\infty}$ is an increasing sequence of nonnegative simple functions such that $\lim_{n\to\infty} f_n(w) I_{A^c}(w) = f(w) I_{A^c}(w) < \infty$ for every $w \in \Omega$. Apply Theorem 1, then

  $$\lim_{n\to\infty} \int f_n I_{A^c} d\nu = \int f I_{A^c} d\nu.$$

  Apply Fact 3 and the above equation gives

  $$\lim_{n\to\infty} \int f_n d\nu = \int f d\nu$$

  since $f_n = f_n I_{A^c}$ $\nu$-a.e. and $f = f I_{A^c}$ $\nu$-a.e.

(ii) Suppose that $\nu(A) > 0$. Then $\int f d\nu \geq \int \infty \cdot I_A d\nu = \infty$.

   * If $\nu(A) = \infty$, then

$$\int f_n d\nu \geq \int 1 \cdot I_{\{f_n > 1\}} d\nu \geq \nu(\{f_n > 1\}),$$

   where $\lim_{n\to\infty} \nu(f_n > 1) = \nu(\cup_{n=1}^{\infty}\{f_n > 1\}) \geq \nu(A) = \infty$ since $A \subset \cup_{n=1}^{\infty}\{f_n > 1\}$. Thus $\lim_{n\to\infty} \int f_n d\nu = \infty = \int f d\nu$.

   * If $0 < \nu(A) < \infty$, then by Fact 4, for $\varepsilon \in (0, \nu(A))$, there exists $B \subset A$ such that $\nu(B) < \varepsilon$ and for $M > 0$, there exists $N$ such that for $n > N$,

$$f_n(w) > M \text{ for } w \in A \cap B^c,$$

   so for $n > N$,

$$\int f_n d\nu \geq \int f_n I_{A\cap B^c} d\nu \geq \int M I_{A\cap B^c} d\nu = M\nu(A \cap B^c).$$

   Therefore,

$$\lim_{n\to\infty} \int f_n d\nu \geq M\nu(A \cap B^c) \text{ for } M > 0.$$

   Since $\nu(A \cap B^c) = \nu(A) - \nu(B) > \nu(A) - \varepsilon > 0$ and $M$ can be made arbitrarily large, we have $\lim_{n\to\infty} \int f_n d\nu = \infty = \int f d\nu$.

The proof of Theorem 1 for the extension case is complete.

- Approximation of a nonnegative function using simple functions. Suppose that $f$ is a nonnegative function defined on $\Omega$. For $n \in \{1, 2, \ldots\}$, and $w \in \Omega$, define

$$f_n(w) = \begin{cases} n & \text{if } f(w) \geq n; \\ \frac{k}{2^n} & \text{if } f(w) \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right), k \in \{0, 1, \ldots, n2^n - 1\}. \end{cases}$$

  Then $f_n \leq f_{n+1}$ on $\Omega$ for every $n$ and $\lim_{n\to\infty} f_n(w) = f(w)$ for $w \in \Omega$.