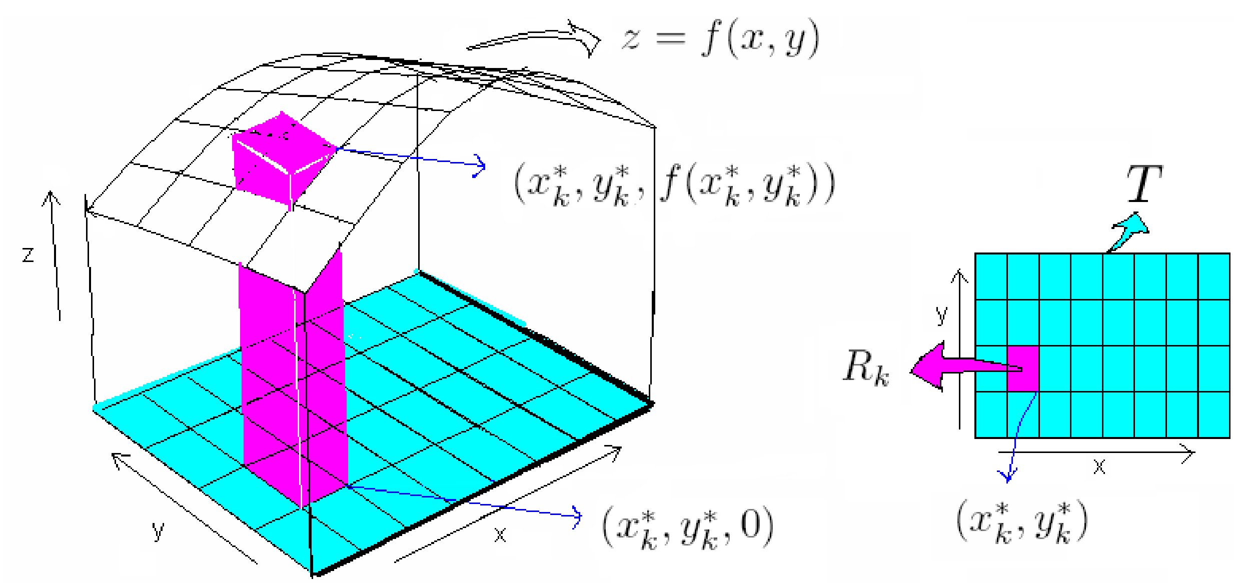
Integration for bivariate functions

- Suppose that $T \subset R^2$ and $f(x, y) \geq 0$ for $(x, y) \in T$. Let S be the region under the surface $z = f(x, y)$ when $(x, y) \in T$. Then the volume (體積) of S is represented by
$$\int_T f(x, y) d(x, y).$$
- Volume approximation.
  - Divide T into N sub-regions R_1, …, R_N, and approximate f by $f(x_k^*, y_k^*)$ on R_k, where $(x_k^*, y_k^*) \in R_k$ is called a sub-region representative (子區域的代表點). Then the volume under the surface $z = f(x, y)$ can be approximated by the Riemann sum (黎曼和)
$$\sum_{k=1}^{N} f(x_k^*, y_k^*) A(R_k),$$
where $A(R_k)$ is the area of R_k.
  - $R_1, \ldots, R_N$ forms a partition (分割) of T, let $\mathcal{P}$ denote the partition. The norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is the maximum of $A(R_1)$, …, $A(R_N)$.



- Definition of $\int_T f(x, y) d(x, y)$.
$$\int_T f(x, y) d(x, y) = \lim_{\|\mathcal{P}\| \to 0} \sum_{k=1}^{N} f(x_k^*, y_k^*) A(R_k),$$

if the limit exists. In such case, we say f is Riemann integrable on T.

- Suppose that $T = [a, b] \times [c, d] = \{(x, y) : x \in [a, b] \text{ and } y \in [c, d]\}$, and f is continuous on T except on a set of area zero. Then f is Riemann integrable on T and

$$\int_T f(x,y)d(x,y) = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

- Example 1. Find the volume of the region $\{(x, y, z) : (x, y) \in [0, 1] \times [0, 2] \text{ and } 0 \le z \le x^3 y\}$.

  Sol. The volume is

$$\begin{aligned} &\int_{[0,1]\times[0,2]} x^3 y d(x,y) \\ &= \int_0^2 \int_0^1 x^3 y dxdy \\ &= \int_0^2 \left(\frac{yx^4}{4} \bigg|_0^1 \right) dy \\ &= \int_0^2 \frac{y}{4} dy = \frac{y^2}{8} \bigg|_0^2 = \frac{1}{2} \end{aligned}$$

- Example 2. Find $\int_{[0,1]\times[3,4]} (x^2 + xy)d(x, y)$.

  Sol.

$$\begin{aligned} \int_{[0,1]\times[3,4]} (x^2+xy)d(x,y) &= \int_0^1 \int_3^4 (x^2+xy)dydx \\ &= \int_0^1 \left(x^2 + \frac{7x}{2} \right) dx = \frac{25}{12}. \end{aligned}$$

- Compuation of $\int_D f(x, y)d(x, y)$ for a general region D.

  – Define

$$g(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D; \\ 0 & \text{otherwise,} \end{cases}$$

  and find a rectangle T such that $T \supset D$, then

$$\int_D f(x,y)d(x,y) = \int_T g(x,y)d(x,y).$$

- Example 3. Let $D = \{(x, y) : x^2 + y^2 \le 1, x \ge 0, y \ge 0\}$, find $\int_D (x^2 + xy)d(x, y)$.

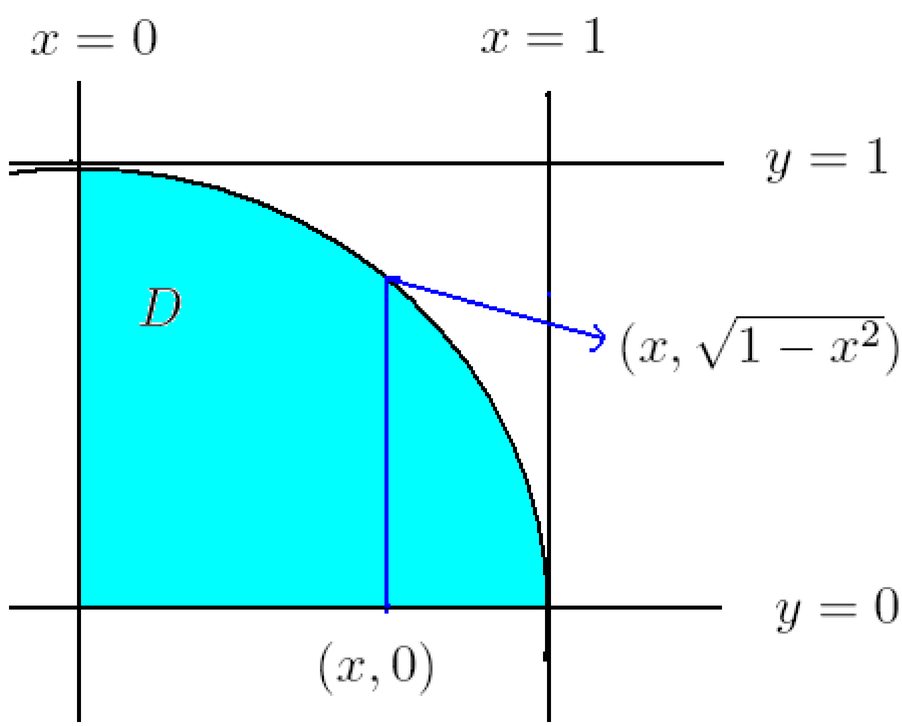
  Sol. Define

$$g(x,y) = \begin{cases} x^2+xy & \text{if } (x,y) \in D; \\ 0 & \text{otherwise.} \end{cases}$$

Since $D \subset [0,1] \times [0,1]$, we have

$$\begin{aligned}
\int_D (x^2 + xy) d(x,y) &= \int_{[0,1]\times[0,1]} g(x,y) d(x,y) \\
&= \int_0^1 \int_0^1 g(x,y) dy dx \\
&= \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + xy) dy dx \\
&= \int_0^1 \left(x^2\sqrt{1-x^2} + \frac{x(1-x^2)}{2} \right) dx \\
&= \frac{\pi}{16} + \frac{1}{8}.
\end{aligned}$$

To find the exact integration region, it helps to use a graph.



- Note that for a region $D = \{(x,y) : a \le x \le b \text{ and } g(x) \le y \le h(x)\}$, where g and h are continuous functions on $[a,b]$, we have

$$\int_D f(x,y) d(x,y) = \int_a^b \int_{g(x)}^{h(x)} f(x,y) dy dx$$

if f is integrable on D. Similarly, for a region $E = \{(x, y) : c \le y \le d \text{ and } f_1(y) \le x \le f_2(y)\}$, where f_1 and f_2 are continuous on $[c, d]$, we have

$$\int_E f(x,y)d(x,y) = \int_c^d \int_{f_1(y)}^{f_2(y)} f(x,y)dxdy$$

if f is integrable on E.

- Some properties of integration (it is assumed that the functions f and g are integrable)
  - Linearity: for constants a and b,

  $$\int_D (af(x,y)+bg(x,y))d(x,y) = a\int_D f(x,y)dxdy + b\int_D g(x,y)d(x,y).$$

  - Dominance rule:

  $$\int_D f(x,y)d(x,y) \le \int_D g(x,y)d(x,y) \text{ if } f \le g.$$

  - Subdivision rule: suppose a region D can be divided into D_1 and D_2 such that $D_1 \cup D_2 = D$ and the area of $D_1 \cap D_2$ is zero, then

  $$\int_D f(x,y)d(x,y) = \int_{D_1} f(x,y)d(x,y) + \int_{D_2} f(x,y)d(x,y).$$

  - $\int_D 1d(x,y) = $ area of D.
  - $\int_D f(x,y)d(x,y) = 0$ if the area of D is 0.
- Let $D = \{(x,y) : x - 2y + 2 \ge 0, x + y \le 1 \text{ and } y \ge 0\}$.

Example 4. Let $D = \{(x,y) : x - 2y + 2 \ge 0, x + y \le 1 \text{ and } y \ge 0\}$, $D_1 = D \cap \{(x,y) : x \le 0\}$ and $D_2 = D \cap \{(x,y) : x \ge 0\}$. Suppose that F is a real-valued continuous function on R^2.

(a) Find two real-valued functions f and g such that

$$\int_{D_1} F(x,y)d(x,y) = \int_{-2}^0 \int_0^{f(x)} F(x,y)dydx$$

and

$$\int_{D_2} F(x,y)d(x,y) = \int_0^1 \int_0^{g(x)} F(x,y)dydx$$

(b) Find $\int_D 1d(x,y)$ using the result in Part (a).

Sol.

(a) Draw the graphs of D_1 and D_2, and we can find that D_1 is the region inside the triangle with vertices $(-2,0)$, $(0,1)$ and $(0,0)$ and D_2 is the region inside the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$. Since

$$D_1 = \{(x,y) : -2 \le x \le 0 \text{ and } 0 \le y \le (x+2)/2\},$$

and

$$D_2 = \{(x,y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1-x\}.$$

we can take $f(x) = (x+2)/2$ and $g(x) = 1-x$.

(b) From Part (a), we have

$$\int_{D_1} 1 d(x,y) = \int_{-2}^{0}\int_{0}^{(x+2)/2} dydx = \int_{-2}^{0} \frac{x+2}{2} dx = 1$$

and

$$\int_{D_2} 1 d(x,y) = \int_{0}^{1}\int_{0}^{1-x} 1dydx = \int_{0}^{1} (1-x)dx = 1/2.$$

It is clear that $D = D_1 \cup D_2$ and the area of $D_1 \cap D_2$ is zero. By the subdivision rule, we have $\int_D 1d(x,y) = \int_{D_1} 1d(x,y) + \int_{D_2} 1d(x,y) = 1 + 1/2 = 1.5$.

- Change of variables (換變數).

Fact 1 *Suppose that T is a subset of R^2 and H is a one-to-one function defined on T and takes values in R^2. Let S be the range of H and let*

$$(x(u,v), y(u,v)) = H^{-1}(u,v)$$

for $(u,v) \in S$. Suppose that x and y are differentiable functions with continuous partial derivatives. Then

$$\int_T f(x,y)d(x,y) = \int_S f(x(u,v), y(u,v))|J(u,v)|d(u,v), \tag{1}$$

where the function J is given by

$$J = \text{ determinant of } \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = x_u y_v - y_u x_v.$$

The function J is called the Jacobian, and we also denote it by

$$\frac{\partial(x,y)}{\partial(u,v)}.$$

- Example 5. Let $D = \{(x,y) : 0 < x^2 + y^2 < 4, x \geq 0, y \geq 0\}$. Find

$$\int_D e^{-x^2-y^2} d(x,y)$$

by making the following change of variables: $r = \sqrt{x^2+y^2}$ and θ is determined by

$$\begin{cases} \cos(\theta) = x/\sqrt{x^2+y^2} \\ \sin(\theta) = y/\sqrt{x^2+y^2} \\ 0 \leq \theta < 2\pi \end{cases} \tag{2}$$

Sol. Given $r = \sqrt{x^2+y^2}$ and θ determined by (2), we have $x = r\cos(\theta)$ and $y = r\sin(\theta)$, so the transform H that maps (x,y) to (r,θ) is one-to-one. Let

$$\begin{aligned} S &= \{H(x,y) : (x,y) \in D\} \\ &= \{(\sqrt{x^2+y^2}, \theta) : (x,y) \in D \text{ and } \theta \text{ is determined by (2)}\}, \end{aligned}$$

then $S = (0,2) \times [0, \pi/2]$, so

$$\begin{aligned} \int_D e^{-x^2-y^2} d(x,y) &= \int_S e^{-r^2} |J(r,\theta)| d(r,\theta) \\ &= \int_{(0,2)\times[0,\pi/2]} e^{-r^2} |J(r,\theta)| d(r,\theta), \end{aligned}$$

where the Jacobian $J(r,\theta)$ is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial \theta} = \cos(\theta) r \cos(\theta) - r(-\sin(\theta))\sin(\theta) = r.$$

Therefore,

$$\int_D e^{-x^2-y^2} d(x,y) = \int_0^{\pi/2} \int_0^2 e^{-r^2} |r| dr d\theta = \frac{\pi(1-e^{-4})}{4}.$$

- In (1), the Jacobian $J(u,v)$ is needed to adjust for the area change in the Riemann sum approximation due to the change of variables. Consider the special case where T is a rectangle region. Let $(u(x,y), v(x,y)) = H(x,y)$ for $(x,y) \in T$, then $S = \{(u(x,y), v(x,y)) : (x,y) \in T\}$.
  - Suppose that $\{R_k\}_k$ forms a partition of T and each R_k is a rectangle region with with left lower vertex (x_k, y_k). Then

    $$\int_T f(x,y) d(x,y) \approx \sum_k f(x_k, y_k) A(R_k).$$

  - Let $(u_k, v_k) = (u(x_k, y_k), v(x_k, y_k))$ and $S_k = \{(u(x,y), v(x,y)) : (x,y) \in R_k\}$, then $(u_k, v_k) \in S_k$ and $\{S_k\}_k$ forms a partition of

    $$S = \{(u(x,y), v(x,y)) : (x,y) \in T\}.$$

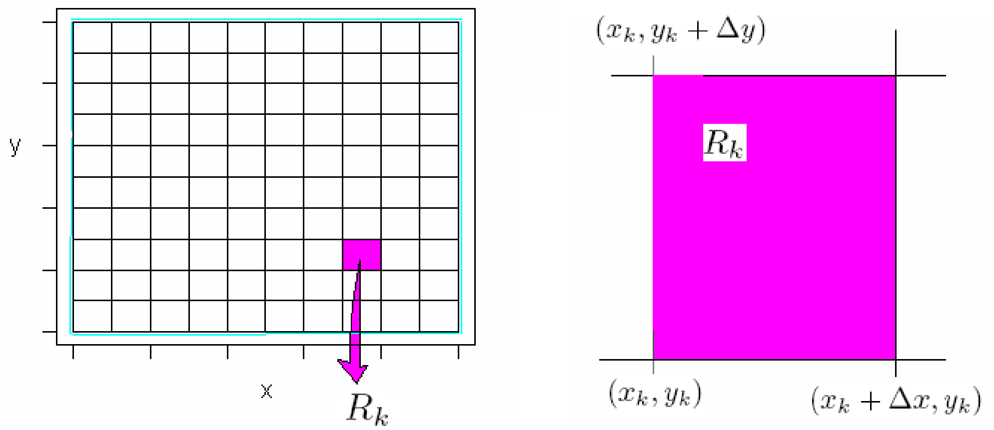
– It can be shown that

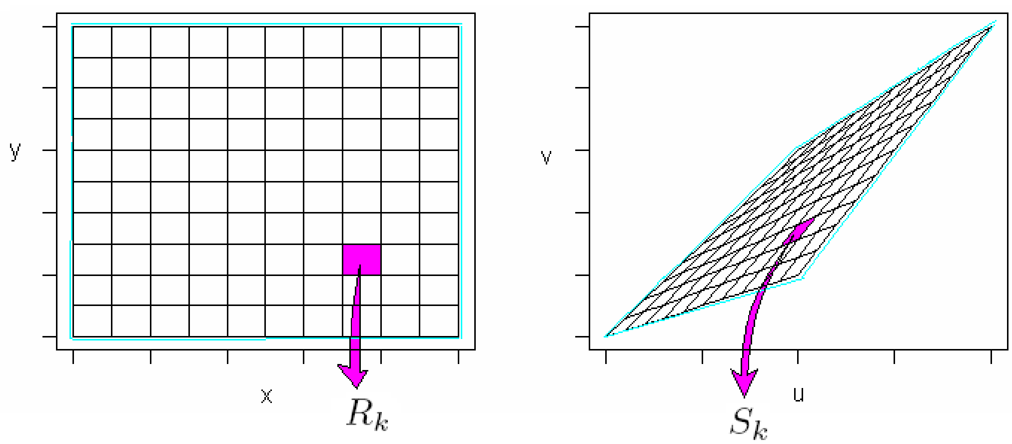
$$\frac{A(R_k)}{A(S_k)} \approx |J(u_k, v_k)|, \tag{3}$$

so

$$\sum_k f(x_k, y_k) A(R_k) \approx \sum_k f(x(u_k, v_k), y(u_k, v_k)) |J(u_k, v_k)| A(S_k),$$

which explains (1) by taking the limits of the Riemann sums.



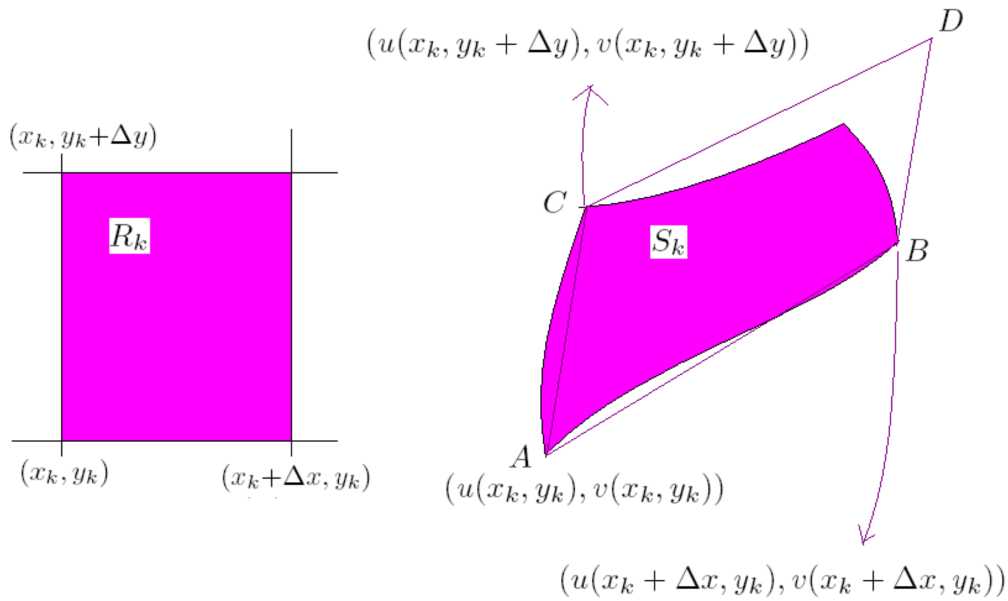


- Justification of (3) : linear approximation and the fact that the area of the parallelogram extended by $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ is given by

$$|a_1 b_2 - a_2 b_1|.$$

- Informal verification of (3).

Let $ABDC$ be the parallelogram in the u-v plane such that $A = (u(x_k, y_k), v(x_k, y_k))$,
$B = (u(x_k + \Delta x, y_k), v(x_k + \Delta x, y_k))$ and $C = (u(x_k, y_k + \Delta y), v(x_k, y_k + \Delta y))$.
Then $A(S_k) \approx$ Area of $ABDC$.



Note that

$$
\begin{aligned}
\overrightarrow{AB} &= (u(x_k + \Delta x, y_k), v(x_k + \Delta x, y_k)) - (u(x_k, y_k), v(x_k, y_k)) \\
&\approx (u_x(x_k, y_k), v_x(x_k, y_k))\Delta x
\end{aligned}
$$

and

$$
\begin{aligned}
\overrightarrow{AC} &= (u(x_k, y_k + \Delta y), v(x_k, y_k + \Delta y)) - (u(x_k, y_k), v(x_k, y_k)) \\
&\approx (u_y(x_k, y_k), v_y(x_k, y_k))\Delta y,
\end{aligned}
$$

so

$$
\begin{aligned}
&\text{Area of } ABDC \\
&\approx \Delta x \Delta y \cdot \text{ the area of the parallelogram extended by } (u_x(x_k, y_k), v_x(x_k, y_k)) \text{ and } (u_y(x_k, y_k), v_y(x_k, y_k)) \\
&= |u_x(x_k, y_k) v_y(x_k, y_k) - u_y(x_k, y_k) v_x(x_k, y_k)| A(R_k).
\end{aligned}
$$

Note that we have used the fact that the area of the parallelogram extended by
$\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ is given by

$$|a_1 b_2 - a_2 b_1|.$$

To see that (3) holds, note that $H(x,y) = (u(x,y), v(x,y)$ for $(x,y) \in T$ and $H^{-1}(u,v) = (x(u,v), y(u,v))$ for $(u,v) \in S = \{H(x,y) : (x,y) \in T\}$, so

$$x(u(x,y), v(x,y)) = x \text{ and } y(u(x,y), v(x,y)) = y$$

for $(x,y) \in T$. Take partial derivatives with respect to x and y, then we have

$$\left(\begin{array}{cc} x_u & x_v \\ y_u & y_v \end{array} \right)\Bigg|_{(u,v)=(u(x,y),v(x,y))} \left(\begin{array}{cc} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{array} \right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right),$$

so

$$J(u_k, v_k)(u_x(x_k, y_k)v_y(x_k, y_k) - u_y(x_k, y_k)v_x(x_k, y_k)) = 1.$$

and (3) holds.

- Example 6. Plot some points in the range of $H(x,y) = (\sqrt{x^2+y^2}, \tan^{-1}(y/x))$ for $(x,y) \in [x_1, x_2] \times [y_1, y_2]$ using the software `R` (can be downloaded from the `R` official site). Run the following `R` commands and we have the plot of some points in the range of H for given x_1, y_1, x_2, y_2.

```
plot.fun <- function(x1, y1, dx, dy){
  x2 <- x1+dx
  y2 <- y1+dy
  m <- 13; n <- 15
  #choose m equally spaced points in [x1,x2] and
  #       n equally spaced points  in [y1,y2]
  # to form m*n points in [x1,x2]x[y1,y2]
  xy.mat <-as.matrix(expand.grid(seq(x1,x2,length=m), seq(y1,y2,length=n)))

  #get the x coordinates and the y coordinates for the m*n points
  #in [x1,x2]x[y1,y2]
  x <- xy.mat[,1]
  y <- xy.mat[,2]

  #plot the m*n points in [x1,x2]x[y1,y2]
  plot(x,y)

  #plot the points (r,theta), where r=sqrt(x^2+y^2), theta = atan(y/x)
  r <- sqrt(x^2+y^2)
  theta <- atan(y/x)
  plot(r,theta)

   #plot the point (r,theta), where r=sqrt(x1^2+y1^2), theta = atan(y1/x1)
  r <- sqrt(x1^2+y1^2)
  theta <- atan(y1/x1)
  points(r, theta, col=2)
}
```

```
#Running plot.fun(x1, y1, dx, dy) gives the plot of some points in the range of H(x,y)
# for (x,y) in [x1, x1+dx] x [y1, y1+dy]
plot.fun(1, 3, 1, 2)
plot.fun(1, 3, 0.01, 0.02)
```

- Consider the $H(x, y)$ in Example 6. Let $(r(x, y), \theta(x, y)) = H(x, y)$. Let A be the point $(r(1, 3), \theta(1, 3))$,

$$B = A + (r_x(1, 3) \cdot 0.01, \theta_x(1, 3) \cdot 0.01),$$

and

$$C = A + (r_y(1, 3) \cdot 0.02, \theta_y(1, 3) \cdot 0.02),$$

then the range of $H(x, y)$ for $(x, y) \in [1, 1.01] \times [3, 3.02]$ can be approximated using the parallelogram extended by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Here

$$\left.\begin{pmatrix} r_x(x, y) & r_y(x, y) \\ \theta_x(x, y) & \theta_y(x, y) \end{pmatrix}\right|_{(x,y)=(1,3)} = \left(\left.\begin{pmatrix} x_r(r, \theta) & x_\theta(r, \theta) \\ y_r(r, \theta) & y_\theta(r, \theta) \end{pmatrix}\right|_{(r,\theta)=H(1,3)} \right)^{-1},$$

where $x(r, \theta) = r\cos(\theta)$ and $y(r, \theta) = r\sin(\theta)$. Run the R commands in Example 6 and then run the following R commands, then we can add the segment $\overline{AB}$ and the segment $\overline{AC}$ in the plot of some points in the range of H for given $x_1 = 1$, $y_1 = 3$, $x_2 = 1.01$, $y_2 = 3.02$.

```
x <- 1; y <- 3
r <- sqrt(x^2+y^2)
theta <- atan(y/x)

#compute M: the matrix of the partial derivatives of x, y with respective to r, theta
x.r <- cos(theta)
x.theta <- r*(-sin(theta))
y.r <- sin(theta)
y.theta <- r*cos(theta)
M <- matrix( c(x.r, y.r, x.theta, y.theta), 2, 2)


#compute the partial derivatives of r, theta with respect to x, y
#using the inverse matrix of M
M1 <- solve(M) #M1 is the inverse matrix of M
r.x <- M1[1,1]
r.y <- M1[1,2]
theta.x <- M1[2,1]
theta.y <- M1[2,2]

#plot the boundaries of the parallelogram
```

```
lines( c(r,r+r.x*0.01), c(theta, theta + theta.x*0.01), col=2)  #AB
lines( c(r,r+r.y*0.02), c(theta, theta + theta.y*0.02), col=3)  #AC
```