Extrema for bivariate functions

- Absolute extrema
  - $f(x, y)$ has an absolute maximum at (x_0, y_0) means $f(x, y) \leq f(x_0, y_0)$ for every (x, y) in the domain of f.
  - $f(x, y)$ has an absolute minimum at (x_0, y_0) means $f(x, y) \geq f(x_0, y_0)$ for every (x, y) in the domain of f.
- Notation.

$$B((x_0, y_0), \delta) = \{(x, y) : \|(x, y) - (x_0, y_0)\| < \delta\}$$

- Relative extrema. Suppose that f is defined on a set containing $B((x_0, y_0), \delta)$ for some $\delta > 0$.
  - $f(x, y)$ has an relative maximum at (x_0, y_0) means $f(x, y) \leq f(x_0, y_0)$ for every (x, y) in $B((x_0, y_0), \delta_0)$ for some $\delta_0 > 0$.
  - $f(x, y)$ has an relative minimum at (x_0, y_0) means $f(x, y) \geq f(x_0, y_0)$ for every (x, y) in $B((x_0, y_0), \delta_0)$ for some $\delta_0 > 0$.
- Partial derivative criteria for relative extrema. Suppose that f is defined on a set containing $B((x_0, y_0), \delta)$ for some $\delta > 0$. If f has a relative maximum or minimum at (x_0, y_0) and f_x and f_y both exist at (x_0, y_0), then

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

  - Proof of Theorem 11.11. Let $g(x) = f(x, y_0)$ and $h(y) = f(x_0, y)$. Suppose that f has a relative maximum (or minimum) at (x_0, y_0), then g has a relative maximum (or minimum) at x_0 and h has a relative maximum (or minimum) at y_0. Since $g'(x_0) = f_x(x_0, y_0)$ and $h'(y_0) = f_y(x_0, y_0)$, $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$ by the critical number theorem.
- If f is differentiable at (x_0, y_0), then

$$f(x, y) \approx \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{L(x,y)}.$$

  The equation for the tangent plant (切平面方程式) to the curve $z = f(x, y)$ at (x_0, y_0) is

$$z = L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

  It is impossible to have $L(x, y) - L(x_0, y_0) \geq 0$ on $B((x_0, y_0), \delta)$ or $L(x, y) - L(x_0, y_0) \leq 0$ on $B((x_0, y_0), \delta)$ unless $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

- Multivariate verison of Taylor's theorem. To obtain the Taylor expansion of $f(x,y)$ at (x_0,y_0), let $g(t) = f(x_0+t(x-x_0), y_0+t(y-y_0))$, then $g(0) = f(x_0,y_0)$ and $g(1) = f(x,y)$. Then the univariate verison of Taylor's theorem says that

$$g(1) = \left(\sum_{k=1}^{n} \frac{g^{(k)}(0)t^k}{k!} \right) + \frac{g^{(n+1)}(c)t^{n+1}}{(n+1)!}, \tag{1}$$

where c is between 1 and 0. (1) gives the multivariate verison of Taylor's theorem.

Example 1. Suppose that f, f_x and f_y are differentiable on $B((x_0,y_0),\delta)$ for some $\delta > 0$ and $(x,y) \in B((x_0,y_0),\delta)$. Then

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &\quad + \frac{1}{2} \begin{pmatrix} x-x_0 & y-y_0 \end{pmatrix} D^2 f(x_*,y_*) \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}, \end{aligned} \tag{2}$$

where

$$D^2 f(x,y) = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix}$$

and the point (x_*,y_*) lies on the segment with endpoints (x_0,y_0) and (x,y).

- Schwarz's theorem states that if f_{xy} and f_{yx} are continuous on $B((x_0,y_0),\delta)$ for some $\delta > 0$, then $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$ (so $D^2 f(x_0,y_0)$ is symmetric in such case).

- Suppose that A is a symmetric $n \times n$ real matrix. A is positive definite (正定) means that

$$v^T A v > 0 \text{ for every non-zero } n \times 1 \text{ vector } v.$$

We use the notation $A > 0$ to indicate that A is positive definite. Note that $A > 0$ if and only if all its eigenvalues are positive.

- Suppose that A is a symmetric $n \times n$ real matrix. A is negatitive definite (負定) means that

$$v^T A v < 0 \text{ for every non-zero } n \times 1 \text{ vector } v.$$

We use the notation $A < 0$ to indicate that A is negatitive definite. Note that $A < 0$ if and only if all its eigenvalues are negative.

- Suppose that $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$ and all the second-order partial derivatives of f are continuous on $B((x_0,y_0),\delta)$ for some $\delta > 0$. Then in (2), $D^2 f(x_*,y_*) \approx D^2 f(x_0,y_0)$, so

$$D^2 f(x_0,y_0) > 0 \Rightarrow \; f \text{ has a relative minimum at } (x_0,y_0),$$

and

$$D^2 f(x_0, y_0) < 0 \Rightarrow \; f \text{ has a relative maximum at } (x_0, y_0).$$

- **Fact 1** *Suppose that A is a real symmetric 2×2 matrix. Then*

  $$A > 0 \Leftrightarrow \textit{det}(A) > 0 \textit{ and } \textit{trace}(A) > 0$$

  and

  $$A < 0 \Leftrightarrow \textit{det}(A) > 0 \textit{ and } \textit{trace}(A) < 0.$$

  Here trace(A) is the sum of all diagonal elements of A.

- A modified second partial test. Suppose that $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ and all the second-order partial derivatives of f are continuous on $B((x_0, y_0), \delta)$ for some $\delta > 0$. Then we have the following results.

  (a) If $\det(D^2 f(x_0, y_0)) > 0$, then

    (i) a relative minimum occurs at (x_0, y_0) if $\text{trace}(D^2 f(x_0, y_0)) > 0$;

    (ii) a relative maximum occurs at (x_0, y_0) if $\text{trace}(D^2 f(x_0, y_0)) < 0$.

  (b) If $\det(D^2 f(x_0, y_0)) < 0$, then there is no relative extremum at (x_0, y_0). In such case, a saddle point (鞍點) occurs at (x_0, y_0).

- Example 2. Suppose that $f(x, y) = x^2 - y^2$. Find all the point(s) where the relative extrema or saddle point(s) of f occur.

  Answer: no relative extrema; saddle point at $(0, 0)$.

  Example 3. Suppose that $f(x, y) = x^2 + y^2 + 2x$. Find all the point(s) where the relative extrema or saddle point(s) of f occur.

  Sol. Compute $f_x(x, y) = 2x + 2$ and $f_y(x, y) = 2y$. Solving $2x + 2 = 0$ and $2y = 0$ gives $(x, y) = (-1, 0)$. Compute the second partial derivatives and we have $f_{xx}(x, y) = 2$, $f_{xy}(x, y) = 0 = f_{yx}(x, y)$ and $f_{yy}(x, y) = 2$, so the matrix

  $$D^2 f(-1, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

  It is clear that $\det(D^2 f(-1, 0)) = 2 \cdot 2 > 0$ and $\text{trace}(D^2 f(-1, 0)) = 2 + 2 > 0$, so $D^2 f(-1, 0) > 0$. From the modified second partial test, f has a relative minimum at $(-1, 0)$ and there is no saddle point.

- Extreme value theorem for a bivariate function. Suppose that f is a bivariate function that is continuous on S, where S is closed (封閉) and bounded (有界). Then f attains both its maximum and minimum on S.

- S is bounded means that $S \subset B((0, 0), M)$ for some $M > 0$.

- S is closed means that S contains all of its boundary points (邊界點).

- A point (x_0, y_0) is a boundary point (邊界點) of S means that for every $\delta > 0$, $B((x_0, y_0), \delta)$ contains both some points S and some points not in S.

- Example. For two closed intervals $[a, b]$ and $[c, d]$, the rectangle $[a, b] \times [c, d] = \{(x, y) : x \in [a, b] \text{ and } y \in [c, d]\}$ is closed and bounded.

- From the extreme value theorem and the partial derivative criteria for relative extrema, the maximum or minimum of f on a closed and bounded set S exists and occurs either at a critical point (points where $f_x = 0 = f_y$ or one of f_x and f_y does not exist) or at a boundary point of S.

Example 4. Find the maximum and minimum of $f(x, y) = x^2 + y$ on $S = \{(x, y) : 0 \le x \le 1 \text{ and } 2 \le y \le 3\}$.

Sol. Since $f_x(x, y) = 2x$ and $f_y(x, y) = 1$, there is no critical point and the maximum and minimum of f occur at boundary points. The boundary points of S are in $S_1 \cup S_2 \cup S_3 \cup S_4$, where

$$S_1 = \{(x, 2) : 0 \le x \le 1\},$$

$$S_2 = \{(x, 3) : 0 \le x \le 1\},$$
$$S_3 = \{(0, y) : 2 \le y \le 3\},$$

and

$$S_4 = \{(1, y) : 2 \le y \le 3\}.$$

On S_1, $f(x, y) = x^2 + 2$ and $0 \le x \le 1$, the maxium and minimum of $x^2 + 2$ for $x \in [0, 1]$ are 3 and 2 respectively. On S_2, $f(x, y) = x^2 + 3$ and $0 \le x \le 1$, the maxium and minimum of $x^2 + 2$ for $x \in [0, 1]$ are 4 and 3 respectively. On S_3, $f(x, y) = y$ and $2 \le y \le 3$, the maxium and minimum of y for $y \in [2, 3]$ are 3 and 2 respectively. On S_4, $f(x, y) = 1+y$ and $2 \le y \le 3$, the maxium and minimum of $1 + y$ for $y \in [2, 3]$ are 4 and 3 respectively. In summary, the maximum of f on S is the maximum of $\{3, 4, 3, 4\}$, which is 4. The minimum of f on S is the minimum of $\{2, 3, 2, 3\}$, which is 2.