**Limit, continuity and differentiation of real-valued functions of several variables.**

- Notation. For $x = (x_1, \ldots, x_d)$ and $a = (a_1, \ldots, a_d)$, define

$$\|x - a\| = \sqrt{(x_1 - a_1)^2 + \cdots + (x_d - a_d)^2}.$$

  - $d = 2$ case. $\|(x, y) - (x_0, y_0)\| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.

- Limit of a real-valued function of $d$ variables. Suppose that $a \in R^d$.

$$\lim_{x \to a} g(x) = L$$

  means that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < \|x - a\| < \delta \Rightarrow |g(x) - L| < \varepsilon.$$

  - $d = 2$ case. $\lim_{(x,y)\to(x_0,y_0)} g(x, y) = L$ means that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < \|(x, y) - (x_0, y_0)\| < \delta \Rightarrow |g(x, y) - L| < \varepsilon$$

- Example 1. Show that $\lim_{(x,y)\to(x_0,y_0)} x = x_0$ for $(x_0, y_0) \in R^2$.
  Sol. Suppose that $(x_0, y_0) \in R^2$. For $\varepsilon > 0$, take $\delta = \varepsilon$, then

$$\begin{aligned} & \|(x, y) - (x_0, y_0)\| < \delta \\ & \Rightarrow |x - x_0| \leq \|(x, y) - (x_0, y_0)\| < \delta \\ & \Rightarrow |x - x_0| < \varepsilon \end{aligned}$$

  so $\lim_{(x,y)\to(x_0,y_0)} x = x_0$ for $(x_0, y_0) \in R^2$.

- It can be shown that $\lim_{(x,y)\to(x_0,y_0)} y = y_0$ for $(x_0, y_0) \in R^2$. The proof is left as an exercise.

- Suppose that $L$ is a real number. Then

$$\lim_{x \to a} g(x) = L \Leftrightarrow \lim_{x \to a} (g(x) - L) = 0 \Leftrightarrow \lim_{x \to a} |g(x) - L| = 0.$$

- The sum/difference/product/quotient/squeeze rules remain valid.

- Example 2. Find $\lim_{(x,y)\to(x_0,y_0)}(x + y)$ for $(x_0, y_0) \in R^2$.
  Sol. For a point $(x_0, y_0) \in R^2$,

$$\lim_{(x,y)\to(x_0,y_0)} (x + y) = \lim_{(x,y)\to(x_0,y_0)} x + \lim_{(x,y)\to(x_0,y_0)} y = x_0 + y_0.$$

  Here we have used the fact that $\lim_{(x,y)\to(x_0,y_0)} y = y_0$ for $(x_0, y_0) \in R^2$, which is left as an exercise.

- Example 3. Find $\lim_{(x,y)\to(1,0)}(x^2+xy+y^2+2x+1)$.

  Sol. $(1^2+1\cdot 0+0^2+2\cdot 1+1)=4$.

- Continuity.

  - $g$ is continuous at $a$ means $\lim\limits_{x\to a} g(x)=g(a)$.
  - $g$ is continuous on $R^2$ means that for every $(x_0,y_0)\in R^2$, $g$ is continuous at $(x_0,y_0)$.

- Example 4. Let $f(x,y)=x$ for $(x,y)\in R^2$. Then $f$ is continuous on $R^2$.

  Note that from Example 1, $\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$ for $(x_0,y_0)\in R^2$, so $f$ is continuous at every point in $R^2$. That is, $f$ is continuous on $R^2$.

- Polynomials of $x$ and $y$ are continuous functions of $(x,y)$ on $R^2$.

- Suppose that $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $f\circ g$ is continuous at $a$.

  - Suppose that $g: R^2\to(-\infty,\infty)$ is continuous on $R^2$ and $f$ is continuous on $(-\infty,\infty)$, then $f\circ g$ is continuous on $R^2$.

- Example 5. Let $f(x,y)=|x|$ for $(x,y)\in R^2$, then $f=g\circ h$, where $h(x,y)=x$ for $(x,y)\in R^2$ and $g(x)=|x|$ for $x\in(-\infty,\infty)$. $f$ is continuous on $R^2$ because $h$ is continuous on $R^2$ and $g$ is continuous on $(-\infty,\infty)$.

- Example 6. Let $g(x,y)=\sin(x+y)$ for $(x,y)\in R^2$, then $g=\sin\circ f$, where $f(x,y)=x+y$. $g$ is continuous on $R^2$ since $f$ is continuous on $R^2$ and $\sin(\cdot)$ is continuous on $(-\infty,\infty)$.

- Example 7. Let $f(x,y)=x^2y/(x^2+y^2)$ for $(x,y)\neq(0,0)$. Show that $\lim_{(x,y)\to(0,0)} f(x,y)=0$ using the inequality $|xy|/(x^2+y^2)\leq 1/2$ for $(x,y)\neq(0,0)$.

  Sol. Since $|xy|/(x^2+y^2)\leq 1/2$ for $(x,y)\neq(0,0)$,

  $$0\leq |f(x,y)|\leq \frac{|x|}{2}$$

  for $(x,y)\neq(0,0)$. From Example 5, $\lim_{(x,y)\to(0,0)}|x|=|0|=0$, so $\lim_{(x,y)\to(0,0)}|x|/2=0=\lim_{(x,y)\neq(0,0)}0$. By squeeze rule,

  $$\lim_{(x,y)\to(0,0)}|f(x,y)|=0,$$

  so $\lim_{(x,y)\to(0,0)} f(x,y)=0$. Here we have used the fact that

  $$\lim_{(x,y)\to(0,0)} f(x,y)=0 \Leftrightarrow \lim_{(x,y)\to(0,0)}|f(x,y)|=0,$$

  which follows from the definition for a limit.

- Composition limit rule. Suppose that $f$ is continuous at $a = (a_1, \ldots, a_d)$, $g_1$, …, $g_d$ are $d$ functions such that $\lim_{t\to t_0} g_i(t) = a_i$ for $i = 1$, …, $d$. Then

$$\lim_{t\to t_0} f(g_1(t), \ldots, g_d(t)) = f\left(\lim_{t\to t_0} g_1(t), \ldots, \lim_{t\to t_0} g_d(t)\right) = f(a).$$

- Example 8. Suppose that $f(x, y) = \dfrac{xy}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$. Show that $f$ is not continuous at $(0, 0)$.

  Sol. Suppose that $f$ is continuous at $(0, 0)$. Then by the composition limit rule, for a constant $k$,

$$\lim_{t\to 0} f(t, kt) = f\left(\lim_{t\to 0} t, \lim_{t\to 0} kt\right) = f(0, 0). \tag{1}$$

  However, from the definition of $f$,

$$\lim_{t\to 0} f(t, kt) = \lim_{t\to 0} \frac{kt^2}{t^2 + k^2t^2} = \frac{k}{1 + k^2}, \tag{2}$$

  so (2) implies that $\lim_{t\to 0} f(t, kt)$ depends on $k$, which contradicts with (1). Therefore, $f$ cannot be continuous at $(0, 0)$.

- Differentiation. Suppose that $f$ is defined at $(x_0, y_0)$ and there exists constants $c$, $d$ and $e$ such that

$$f(x, y) = c + d(x - x_0) + e(y - y_0) + \varepsilon(x, y)\|(x, y) - (x_0, y_0)\|, \tag{3}$$

  where $\lim_{(x,y)\to(x_0,y_0)} \varepsilon(x, y) = 0 = \varepsilon(x_0, y_0)$. Then we say that $f$ is differentiable at $(x_0, y_0)$.

- When $f$ is differentiable at $(x_0, y_0)$, the constants $c$, $d$ and $e$ in (3) can be found as follows.

  - In (3), plug in $(x, y) = (x_0, y_0)$ and we have $c = f(x_0, y_0)$.
  - In (3), plug in $y = y_0$, then

$$\lim_{x\to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} = \lim_{x\to x_0} \frac{d(x - x_0) + \varepsilon(x, y_0)|x - x_0|}{x - x_0} = d.$$

  - In (3), plug in $x = x_0$, then

$$\lim_{y\to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0} = \lim_{y\to y_0} \frac{e(y - y_0) + \varepsilon(x_0, y)|y - y_0|}{y - y_0} = e.$$

- If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

- Partial derivatives (two variable case).

- The partial derivative (偏導數) of $f(x, y)$ with respect to $x$ at $(x_0, y_0)$ is

$$\lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0},$$

which is denoted by $f_x(x_0, y_0)$, $\left.\dfrac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(x_0,y_0)}$, $\left.\dfrac{\partial}{\partial x} f(x,y)\right|_{(x,y)=(x_0,y_0)}$, or $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$.

- The partial derivative of $f(x, y)$ with respect to $y$ at $(x_0, y_0)$ is

$$\lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0},$$

which is denoted by $f_y(x_0, y_0)$, $\left.\dfrac{\partial f(x,y)}{\partial y}\right|_{(x,y)=(x_0,y_0)}$, $\left.\dfrac{\partial}{\partial y} f(x,y)\right|_{(x,y)=(x_0,y_0)}$ or $\left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)}$.

Example 9. $f(x, y) = xy + y^2$. Find $f_x(1, 2)$ and $f_y(1, 2)$.

Sol 1.

$$f_x(1,2) = \lim_{x \to 1} \frac{f(x,2) - f(1,2)}{x - 1} = \left.\frac{d}{dx}(x \cdot 2 + 2^2)\right|_{x=1} = 2,$$

and

$$f_y(1,2) = \lim_{y \to 2} \frac{f(1,y) - f(1,2)}{y - 2} = \left.\frac{d}{dy}(1 \cdot y + y^2)\right|_{y=2} = 1 + 2y|_{y=2} = 5.$$

Sol 2.

$$\begin{aligned} f_x(x_0, y_0) &= \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} \\ &= \left.\frac{d}{dx}(xy_0 + y_0^2)\right|_{x=x_0} \\ &= y_0|_{x=x_0} = y_0, \end{aligned}$$

so $f_x(1, 2) = 2$.

$$f_y(x_0, y_0) = \left.\frac{d}{dy}(x_0 y + y^2)\right|_{y=y_0} = (x_0 + 2y)|_{y=y_0} = x_0 + 2y_0,$$

so $f_y(1, 2) = 1 + 2 \cdot 2 = 5$.

- Note that in Example 9, $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist for every $(x_0, y_0) \in R^2$, so $f_x$ and $f_y$ can be considered as functions defined on $R^2$.

- Notation. $\frac{\partial}{\partial x}f(x,y)$ means $f_x(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ means $f_y(x,y)$.

  Example 10. Find $\frac{\partial}{\partial x}(xy+y^2)$ and $\frac{\partial}{\partial y}(xy+y^2)$.

  Sol.

  $\frac{\partial}{\partial x}(xy+y^2) = y$(treating $y$ as a constant, take derivative with respect to $x$)

  and

  $\frac{\partial}{\partial y}(xy+y^2) = x+2y$(treating $x$ as a constant, take derivative with respect to $y$).

- Notation. $B((x_0, y_0), r) = \{(x, y) : \|(x, y) - (x_0, y_0)\| < r\}$.

- **Fact 1** *If $f$ is defined on $B((x_0, y_0), r)$ for some $r > 0$ and both $f_x$ and $f_y$ are continuous on $B((x_0, y_0), r)$, then $f$ is differentiable at $(x_0, y_0)$. In particular, if $f_x$ and $f_y$ are continuous on $R^2$, then $f$ is differentiable on $R^2$.*

- Suppose that $f$ is differentiable at $(x_0, y_0)$. Then the equation of tangent plane to the surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

- Example 11. Suppose that $f(x,y) = x^2 + xy + x + 3y$. Show that $f$ is differentiable on $R^2$. Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(0, 1, f(0, 1))$.

  Sol.

$$f_x(x,y) = \frac{\partial}{\partial x}(x^2 + xy + x + 3y) = 2x + y + 1$$

  and

$$f_y(x,y) = \frac{\partial}{\partial y}(x^2 + xy + x + 3y) = x + 3$$

  Since both $f_x$ and $f_y$ are polynomials of $x$ and $y$, we have $\lim_{(x,y)\to(x_0,y_0)} f_x(x,y) = f_x(x_0, y_0)$ and $\lim_{(x,y)\to(x_0,y_0)} f_y(x,y) = f_y(x_0, y_0)$ for $(x_0, y_0) \in R^2$, which means that $f_x$ and $f_y$ are continuous on $R^2$. By Fact 1, $f$ is differentiable on $R^2$. The equation of the tangent plane to the surface $z = f(x, y)$ at the point $(0, 1, f(0, 1))$ is $z = f(0, 1) + f_x(0, 1)(x - 0) + f_y(0, 1)(y - 1)$. Direct calculation gives $f(0, 1) = 3$, $f_x(0, 1) = 2$ and $f_y(0, 1) = 3$, so the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(0, 1, f(0, 1))$ is $z = 3 + 2x + 3(y - 1)$.

- Example 12. Find the equation of tangent plane to the surface $z = x^2 + 2x + \sin(xy) + 3y^2 + 2$ at $(-1, 0, 1)$.

  Sol. Since

  $$\frac{\partial}{\partial x}(x^2 + 2x + \sin(xy) + 3y^2 + 2)\Big|_{(x,y)=(-1,0)} = (2x + 2 + \cos(xy)y)|_{(x,y)=(-1,0)} = 0$$

  and

  $$\frac{\partial}{\partial y}(x^2 + 2x + \sin(xy) + 3y^2 + 2)\Big|_{(x,y)=(-1,0)} = (\cos(xy)x + 6y)|_{(x,y)=(-1,0)} = -1,$$

  the equation of tangent plane to the surface $z = x^2 + 2x + \sin(xy) + 3y^2$ at $(-1, 0, 1)$ is $z = 1 + 0 \cdot (x - (-1)) + (-1)(y - 0)$, which is $z = 1 - y$.

- The existence of partial derivatives does not imply differentiability.

  Example 13. Suppose that $f(x, y) = 1$ for $x > 0$ and $y > 0$ and $f(x, y) = 0$ otherwise. Then $f_x(0, 0) = 0 = f_y(0, 0)$. However, $f$ is not differentiable at $(0, 0)$.

- Partial derivatives (three variable case).
  - The partial derivative of $f(x, y, z)$ with respect to $x$ at $(x_0, y_0, z_0)$ is

    $$\lim_{x \to x_0} \frac{f(x, y_0, z_0) - f(x_0, y_0, z_0)}{x - x_0},$$

    which is denoted by $f_x(x_0, y_0, z_0)$ or

    $$\frac{\partial}{\partial x} f(x, y, z)\Big|_{(x,y,z)=(x_0,y_0,z_0)} \quad \text{or} \quad \frac{\partial f}{\partial x}\Big|_{(x_0,y_0,z_0)}.$$

  - The partial derivative of $f(x, y, z)$ with respect to $y$ at $(x_0, y_0, z_0)$ is

    $$\lim_{y \to y_0} \frac{f(x_0, y, z_0) - f(x_0, y_0, z_0)}{y - y_0},$$

    which is denoted by $f_y(x_0, y_0, z_0)$ or

    $$\frac{\partial}{\partial y} f(x, y, z)\Big|_{(x,y,z)=(x_0,y_0,z_0)} \quad \text{or} \quad \frac{\partial f}{\partial y}\Big|_{(x_0,y_0,z_0)}.$$

  - The partial derivative of $f(x, y, z)$ with respect to $z$ at $(x_0, y_0, z_0)$ is

    $$\lim_{z \to z_0} \frac{f(x_0, y_0, z) - f(x_0, y_0, z_0)}{z - z_0},$$

    which is denoted by $f_z(x_0, y_0, z_0)$ or

    $$\frac{\partial}{\partial z} f(x, y, z)\Big|_{(x,y,z)=(x_0,y_0,z_0)} \quad \text{or} \quad \frac{\partial f}{\partial z}\Big|_{(x_0,y_0,z_0)}.$$

- General version of Fact 1. Suppose that all the (first order) partial derivative functions of $f$ exist on $B(a, \delta)$ for some $\delta > 0$ and are continuous at $a$, where $B(a, \delta) = \{x : \|x - a\| < \delta\}$. Then $f$ is differentiable at $a$.

- Higher order partial derivatives. For a function $f(x, y)$, $f_x$ and $f_y$ are called the first order partial derivatives of $f$. The four partial derivatives of $f_x$ and $f_y$ are called the second order partial derivatives of $f$. Below are their definitions and various expressions:

$$f_{xx}(x,y) = \frac{\partial}{\partial x} f_x(x,y) = \frac{\partial^2}{\partial x^2} f(x,y),$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y} f_x(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y),$$

$$f_{yx}(x,y) = \frac{\partial}{\partial x} f_y(x,y) = \frac{\partial^2}{\partial x \partial y} f(x,y),$$

and

$$f_{yy}(x,y) = \frac{\partial}{\partial y} f_y(x,y) = \frac{\partial^2}{\partial y^2} f(x,y).$$

The third order partial derivatives of $f$ are the eight partial derivatives of the second order partial derivatives. Higher order partial derivatives can be defined accordingly. For example,

$$f_{xxy}(x,y) = \frac{\partial}{\partial y} f_{xx}(x,y) = \frac{\partial^3}{\partial y \partial x^2} f(x,y)$$

and

$$f_{xxyx}(x,y) = \frac{\partial}{\partial x} f_{xxy}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial^3}{\partial y \partial x^2} f(x,y)\right) = \frac{\partial^4}{\partial x \partial y \partial x^2} f(x,y).$$

- For a function $f(x, y, z)$, the higher order partial derivatives can be also defined in a similar way. For instance,

$$f_{zxy}(x,y,z) = \frac{\partial}{\partial y} f_{zx}(x,y,z) = \frac{\partial^2}{\partial y \partial x} f_z(x,y,z) = \frac{\partial^3}{\partial y \partial x \partial z} f(x,y,z).$$

Example 14. $f(x, y) = x^2y + y^2$. Find $f_x$, $f_y$, $f_{xx}$, $f_{xy}$, $f_{yx}$, $f_{yy}$, and $f_{xyy}$.

Answers. $f_x(x, y) = 2xy$, $f_y(x, y) = x^2 + 2y$, $f_{xx}(x, y) = 2y$, $f_{xy}(x, y) = 2x = f_{yx}(x, y)$, $f_{yy}(x, y) = 2$ and $f_{xyy}(x, y) = 0$.

Example 15. Suppose that $f(x, y, z) = x + yz + \sin(x^2z)$. Find $f_x$ and $f_{xzy}$.

Answers: $f_x(x, y, z) = 1 + 2zx\cos(x^2z)$. $f_{xzy}(x, y, z) = 0$.

- Chain rule - one parameter version. Suppose that $g(t) = f(x_1(t), \ldots, x_d(t))$, where $x_1(t)$, …, $x_d(t)$ are differentiable function of $t$ and $f$ is differentiable on the range of $(x_1(t), \ldots, x_d(t))$, then

$$g'(t) = \sum_{j=1}^{d} \frac{\partial f}{\partial x_j}(x_1(t), \ldots, x_d(t))x_j'(t). \tag{4}$$

(4) is sometimes expressed as

$$\frac{dg}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_d}\frac{dx_d}{dt}.$$

  - When $d = 2$ and $g(t) = f(x(t), y(t))$, where $x$ and $y$ are differentiable at $t$ and $f$ is differnetiable at $(x(t), y(t))$, (4) becomes

$$g'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t). \tag{5}$$

  - Proof of (5). Recall that $f$ is differentiable at $(x_0, y_0)$ means that

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &\quad +\varepsilon(x,y)\|(x-x_0, y-y_0)\|, \end{aligned} \tag{6}$$

    where $\lim_{(x,y)\to(x_0,y_0)} \varepsilon(x,y) = 0 = \varepsilon(x_0,y_0)$. Apply (6) with $(x, y) = (x(t+h), y(t+h))$ and $(x_0, y_0) = (x(t), y(t))$ to find $g'(t)$.

Example 16. Suppose that $f$ is a function on $R^2$ such that $f_x(x, y) = 2x + y$ and $f_y(x, y) = 2y + x$. Let $h(t) = f(t, t^2)$ for $t \in R$. Find $h'(t)$.

Sol.

$$h'(t) = f_x(t,t^2)\frac{d}{dt}t + f_y(t,t^2)\frac{d}{dt}t^2 = 2t + t^2 + 2t(2t^2 + t) = 4t^3 + 3t^2 + 2t$$

- Chain rule - two parameter version.
  - Suppose that $z(u, v) = f(x, y)$, where $x = x(u, v)$ and $y = y(u, v)$. Suppose that $x_u, x_v, y_u$ and $y_v$ exist and $f$ is differentiable. Then

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \text{ and } \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

  - Suppose that $u = f(x, y, z)$, where $x = x(s, t), y = y(s, t)$ and $z = z(s, t)$. Suppose that the partial derivatives of $x, y$ and $z$ with respect to $s$ and $t$ exist and $f$ is diffferentiable. Then

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

    and

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$