Power series and Taylor series

- A power series (冪級數) in $x-c$ is a series of the form

$$\sum_{k=0}^{\infty} a_k (x-c)^k.$$

The a_k's are called the coefficients of the power series (冪級數的係數) .

- Radius of convergence (收斂半徑) . For a power series $\sum_{k=0}^{\infty} a_k(x-c)^k$, one of (i)-(iii) holds.

  (i) The series converges absolutely for every real number x.

  (ii) The series converges only for $x = c$.

  (iii) There exists a number $r \in (0, \infty)$ such that the series converges absolutely for $|x-c| < r$ and diverges for $|x-c| > r$.

We say that the radius of convergence of the power series is

$$\begin{cases} \infty & \text{in Case (i);} \\ 0 & \text{in Case (ii);} \\ r & \text{in Case (iii).} \end{cases}$$

- Example 1. Let r be the radius of convergence for a power series $\sum_{k=0}^{\infty} a_k(x-2)^k$. What can be said about r in each of the following cases?

  (a) The power series converges for $x = 2.1$.

  (b) The power series diverges for $x = 3.1$.

  (c) The power series converges for $x = 3$ and diverges for $x = 1$.

Sol. Note that the power series converges absolutely for $|x-2| < r$ and diverges for $|x-2| > r$, so $r \geq |2.1-2| = 0.1$ in Case (a) and $r \leq |3.1-2| = 1.1$ in Case (b). In Case (c), since $r \geq |3-2| = 1$ and $r \leq |1-2| = 1$, we have $r = 1$.

- Ratio test (or root test) can be used for finding the radius convergence of a power series.

Example 2. Find the radius of convergence of $\sum_{k=0}^{\infty} x^k/k!$.

Sol. We can show that the series $\sum_{k=0}^{\infty} |x^k|/k!$ converges for all x using the ratio test. Therefore, the radius of convergence for $\sum_{k=0}^{\infty} |x^k|/k!$ is ∞.

Example 3. Find the radius of convergence of the power series $\sum_{k=1}^{\infty} x^k/\sqrt{k}$.

Sol. From the ratio test, the series $\sum_{k=1}^{\infty} |x^k|/\sqrt{k}$ converges when $|x| < 1$ and diverges when $|x| > 1$. Since the series $\sum_{k=1}^{\infty} x^k/\sqrt{k}$ does not diverge for $|x| < 1$, its radius of convergence cannot be less than 1. Also, the series $\sum_{k=1}^{\infty} x^k/\sqrt{k}$ does not converge absolutely for $|x| > 1$, so its radius of convergence cannot be greater than 1. Therefore, the radius of convergence is equal to 1.

- Term by term differentitation and integration. Suppose that a power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ has radius of convergence $r > 0$. Then the following results hold.

  - The series $\sum_{k=1}^{\infty} a_k k(x-c)^{k-1}$ has radius of convergence r, and

$$\frac{d}{dx}\sum_{k=0}^{\infty} a_k(x-c)^k = \sum_{k=1}^{\infty} a_k k(x-c)^{k-1} \text{ for } |x-c| < r.$$

  - The series $\displaystyle\sum_{k=0}^{\infty} \frac{a_k(x-c)^{k+1}}{k+1}$ has radius convergence r, and

$$\frac{d}{dx}\sum_{k=0}^{\infty} \frac{a_k(x-c)^{k+1}}{k+1} = \sum_{k=0}^{\infty} a_k(x-c)^k \text{ for } |x-c| < r.$$

Example 4. Let $f(x) = \sum_{k=0}^{\infty} x^k/k!$ for $x \in (-\infty, \infty)$. Show that $f'(x) = f(x)$ for $x \in (-\infty, \infty)$.

Sol. Term by term differentiation.

- Example 5. Show that $\ln(1+x) = \sum_{k=0}^{\infty}(-1)^k x^{k+1}/(k+1)$ for $|x| < 1$.
Sol. Since $1/(1+x) = \sum_{k=0}^{\infty}(-x)^k$ and the power series $\sum_{k=0}^{\infty}(-x)^k$ has radius of convergence 1, from term by term integration, we have

$$\frac{d}{dx}\sum_{k=0}^{\infty}(-1)^k\frac{x^{k+1}}{k+1} = \frac{1}{1+x}$$

and

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty}(-1)^k\frac{x^{k+1}}{k+1} - \ln(1+x)\right) = 0$$

for $|x| < 1$. From the zero derivative theorem, $\ln(1+x) = C + \sum_{k=0}^{\infty}(-1)^k x^{k+1}/(k+1)$ for $|x| < 1$. When $x = 0$, we have $\ln(1+0) = C + 0$, so $C = 0$ and $\ln(1+x) = \sum_{k=0}^{\infty}(-1)^k x^{k+1}/(k+1)$ for $|x| < 1$.

- Uniqueness theorem (Theorem 8.24). Suppose that $\displaystyle\sum_{k=0}^{\infty} a_k(x-c)^k$ has radius of convergence $r > 0$. Let $f(x) = \displaystyle\sum_{k=0}^{\infty} a_k(x-c)^k$ for $|x-c| < r$. Then $a_k = f^{(k)}(c)/k!$ for all k.

- The series $\sum_{k=0}^{\infty} f^{(k)}(c)(x-c)^k/k!$ is called the Taylor series (泰勒級數) of f at c. When $c=0$, it is called the Maclaurin series (麥克勞倫級數) of f. The partial sum of the first $n+1$ terms of the Maclaurin series is denoted by $M_n(x)$.

- Note that if $f(x)=\sum_{k=0}^{\infty} a_k(x-c)^k$ for $|x-c|<r$, where $r>0$ is the radius of convergence of $\sum_{k=0}^{\infty} a_k(x-c)^k$, then by the uniqueness theorem, the Taylor series of f at c is $\sum_{k=0}^{\infty} a_k(x-c)^k$.

  Example 6. Find the Maclaurin series of $1/(1+x^2)$.

  Sol. Since $1/(1+x^2)=\sum_{m=0}^{\infty}(-1)^m x^{2m}$ for $|x|<1$, the Maclaurin series of $1/(1+x^2)$ is $\sum_{m=0}^{\infty}(-1)^m x^{2m}$.

- It is possible that $f(x)\neq\sum_{k=0}^{\infty} f^{(k)}(c)(x-c)^k/k!$ for $|x-c|<r$, where r is the radius of convergence of the Taylor series $\sum_{k=0}^{\infty} f^{(k)}(c)(x-c)^k/k!$. If $f(x)=\sum_{k=0}^{\infty} f^{(k)}(c)(x-c)^k/k!$, then we say that f can be represented by its Taylor series at c.

  Example 7. Let $f(x)=e^{-1/x^2}$ for $x\neq 0$ and $f(0)=0$. Show that f cannot be represented by its Maclaurin series.

  Sol. It can be shown that for $k\geq 1$,

$$f^{(k)}(x)=\begin{cases} e^{-1/x^2}p_k(1/x) & \text{if } x\neq 0;\\ 0 & \text{if } x=0,\end{cases}$$

  where $p_k(1/x)$ is a polynomial of $1/x$. Therefore, the Maclaurin series of f is $\sum_{k=0}^{\infty} f^{(k)}(0)x^k/k! = 0\neq f(x)$ for $x\neq 0$.

  Example 8. Find the Maclaurin series for $\cos(x)$.

  Ans. The Maclaurin series for $\cos(x)$ is $\displaystyle\sum_{k=0}^{\infty}\frac{b_k}{k!}x^k$, where

$$b_k=\frac{d^k}{dx^k}\cos(x)\bigg|_{x=0}=\begin{cases} 0 & \text{if } k \text{ is odd;}\\ (-1)^{k/2} & \text{if } k \text{ is even.}\end{cases}$$

  We can also express the Maclaurin series for $\cos(x)$ as $\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^{2m}$.

  Example 9. Find the Taylor series of $\ln(x)$ at 1.

  Ans. The Taylor series of $\ln(x)$ at 1 is $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(x-1)^k$.

- Taylor's theorem (Theorem 8.25). Suppose that $f^{(k)}$ exists for $k \leq (n+1)$ on an open interval I that contains a number c. Then for $x \in I$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{n+1}(z_n)}{(n+1)!}(x-c)^{n+1}, \qquad (1)$$

where z_n is a number between c and x.

Example 10. Find the Maclaurin polynomial series $M_5(x)$ for e^x and use this polynomial to approximate e. Use the Taylor's theorem to determine the approximation accuracy.

Sol. $M_5(x) = \sum_{k=0}^{5} x^k/k!$. From Taylor's theorem, $e^1 - M_5(1) = e^z 1^6/6!$, where z is between 0 and 1. Therefore, $|e - M_5(1)| \leq e/6! \leq 3/6! = 1/240$. Here $e \leq 3$ follows from the fact that $(1+1/n)^n \leq \sum_{k=0}^{n} 1/k! \leq 1 + \sum_{k=1}^{n} 2^{-k+1} \leq 3$.

Example 11. Show that $\cos(x)$ can be represented by its Maclaurin series.

Sol. Apply Taylor's theorem with $f(x) = \cos(x)$ and show that the remainder term $\dfrac{f^{n+1}(z_n)}{(n+1)!}(x-0)^{n+1}$ converges to 0 as $n \to \infty$.

- From Taylor's theorem, if f'' exists on an open interval containing $[a,b]$ and

$$|f''(x)| \leq M_0 \text{ for } x \in [a,b],$$

then for $c \in (a,b)$,

$$|f(x) - f(c) - f'(c)(x-c)| \leq \frac{M_0(x-c)^2}{2} \text{ for } x \in [a,b].$$

Example 12. Suppose that $f(x) = \sin(x)$.

(a) Find the maximum of $|f''|$ on $[-0.05, 0.05]$.

(b) Find a, b and M such that $|f(x) - a - bx| \leq Mx^2$ for $|x| \leq 0.05$.

Answers: (a) $\sin(0.05)$. (b) $a = f(0) = 0$, $b = f'(0) = 1$ and $M = 0.025$ (M can be any number such that $M \geq 0.5\sin(0.05) = 0.024989...$).

Proof for Taylor's theorem (Theorem 8.25).

We will first show that

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^{m-1} \int_c^x \frac{f^{(m)}(t)}{(m-1)!}(t-x)^{m-1}dt \tag{2}$$

for $1 \leq m \leq n$. It is clear that (2) holds for $m = 1$. Suppose that (2) holds for a particular $m \leq n$ and suppose that $m+1 \leq n$, then

$$\begin{aligned}
f(x) &= \sum_{k=0}^{m-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^{m-1} \int_c^x f^{(m)}(t) d\frac{(t-x)^m}{m!} \\
&= \sum_{k=0}^{m-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^{m-1}\ f^{(m)}(t)\frac{(t-x)^m}{m!}\bigg|_c^x \\
&\quad -(-1)^{m-1} \int_c^x \frac{(t-x)^m}{m!} f^{(m+1)}(t)dt \\
&= \sum_{k=0}^{m-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^m f^{(m)}(c)\frac{(c-x)^m}{m!} \\
&\quad +(-1)^m \int_c^x \frac{f^{(m+1)}(t)}{m!}(t-x)^m dt,
\end{aligned}$$

so (2) holds for $m+1$. By induction, (2) holds for $1 \leq m \leq n$.

Note that (1) follows from (2) with $m = n$ and the fact that

$$(-1)^{n-1} \int_c^x \frac{f^{(n)}(t)}{(n-1)!}(t-x)^{n-1}dt - \frac{f^{(n)}(c)}{n!}(x-c)^n = \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-c)^{n+1} \tag{3}$$

for some z_n between x and c. Now we are ready to prove (3). Let J be the closed interval with endpoints c and x. Define

$$g(t) = \frac{f^{(n)}(t) - f^{(n)}(c)}{t-c} \text{ for } t \neq c$$

and $g(c) = f^{(n+1)}(c)$, then g is continuous on J. Let M_g and m_g be the maximum and minimum of g on J respectively. Let

$$\begin{aligned}
I &= (-1)^{n-1} \int_c^x \frac{f^{(n)}(t)}{(n-1)!}(t-x)^{n-1}dt - \frac{f^{(n)}(c)}{n!}(x-c)^n \\
&= (-1)^{n-1} \int_c^x \left(f^{(n)}(t) - f^{(n)}(c)\right) \frac{(t-x)^{n-1}}{(n-1)!}dt \\
&= \int_c^x \left(f^{(n)}(t) - f^{(n)}(c)\right) \frac{(x-t)^{n-1}}{(n-1)!}dt,
\end{aligned}$$

then

$$\begin{aligned}
\frac{I}{(x-c)^{n+1}} &= \int_c^x \frac{f^{(n)}(t)-f^{(n)}(c)}{t-c}\frac{(t-c)(x-t)^{n-1}}{(n-1)!(x-c)^{n+1}}dt \\
&= \int_0^1 g(c+u(x-c))\frac{u(1-u)^{n-1}}{(n-1)!}du
\end{aligned}$$

Since $c+u(x-c)\in J$ for $u\in[0,1]$, we have $m_g \le g(c+u(x-c)) \le M_g$ and

$$\int_0^1 m_g\frac{u(1-u)^{n-1}}{(n-1)!}du \le \frac{I}{(x-c)^{n+1}} \le \int_0^1 M_g\frac{u(1-u)^{n-1}}{(n-1)!}du,$$

so

$$m_g \le \frac{I}{(x-c)^{n+1}/(n+1)!} \le M_g.$$

By the intermediate value theorem, there exists c_1 in J such that

$$g(c_1) = \frac{I}{(x-c)^{n+1}/(n+1)!}.$$

From the definition of g, there exists z_n between c and c_1 such that $g(c_1) = f^{(n+1)}(z_n)$, so

$$f^{(n+1)}(z_n) = \frac{I}{(x-c)^{n+1}/(n+1)!},$$

and (3) holds. The proof of (1) is complete.