Series convergence tests

- Suppose that $a_k \geq 0$ for $k \geq m$. Then the series $\sum_{k=m}^{\infty} a_k$ converges if the partial sum sequence $\{\sum_{k=m}^{n} a_k\}$ is bounded above. If $\{\sum_{k=m}^{n} a_k\}$ is not bounded above, then $\sum_{k=m}^{\infty} a_k = \infty$ diverges.

- Integral test. Suppose that $f \geq 0$ is decreasing and continuous on $[m, \infty)$, where m is a nonnegative integer. Then $\sum_{k=m}^{n} f(k)$ and $\int_m^{\infty} f(x)dx (= \lim_{n\to\infty} \int_m^n f(x)dx)$ both converge or both diverge. This result can be expressed as the statements in (i) and (ii):

  (i) If $\lim_{n\to\infty} \int_m^n f(x)dx$ exists, then $\sum_{k=m}^{\infty} f(k)$ converges.

  (ii) If $\lim_{n\to\infty} \int_m^n f(x)dx = \infty$, then $\sum_{k=m}^{\infty} f(k) = \infty$.

  We can prove (i) and (ii) using the following result:

  $$f(k+1) \leq \int_k^{k+1} f(x)dx \leq f(k) \text{ for } k \geq m.$$

  Example 1. Show that $\sum_{k=1}^{\infty} k^{-p}$ is convergent if $p > 1$ and is divergent if $0 < p \leq 1$.

  Sol. Apply the integral test with $f(x) = x^{-p}$.

- $\sum_{k=1}^{\infty} k^{-p}$ is called a p-series.

- Ratio test and root test. Suppose that $a_k \geq 0$ for all $k \geq m$.

  - Ratio test. Suppose that $\lim_{k\to\infty} a_{k+1}/a_k = L$. If $L < 1$, then $\sum_{k=m}^{\infty} a_k$ converges. If $L > 1$, then $\sum_{k=m}^{\infty} a_k$ diverges.
  - Root test. Suppose that $\lim_{k\to\infty} (a_k)^{1/k} = L$. If $L < 1$, then $\sum_{k=m}^{\infty} a_k$ converges. If $L > 1$, then $\sum_{k=m}^{\infty} a_k$ diverges.
  - The proofs for ratio test and root test are based on direct comparison test (to be introduced later).

  Example 2. Show that $\sum_{k=1}^{\infty} (k!)^{-1}$ converges.

  Sol. Use ratio test.

  Example 3. Show that $\sum_{k=1}^{\infty} (k)^{-k}$ converges.

  Sol. Use root test.

- Direct comparison test.

  (i) Suppose that $0 \le a_k \le c_k$ for $k \ge m$ and $\sum_{k=m}^{\infty} c_k$ converges. Then $\sum_{k=m}^{\infty} a_k$ converges.

  (ii) Suppose that $0 \le c_k \le a_k$ for $k \ge m$ and $\sum_{k=m}^{\infty} c_k = \infty$. Then $\sum_{k=m}^{\infty} a_k$ diverges ($= \infty$).

Example 4. Show that $\sum_{k=1}^{\infty} k^{-1}2^{-k}$ converges.

Sol. Compare $k^{-1}2^{-k}$ with 2^{-k}.

- Limit comparison test. Suppose that $a_k \ge 0$ and $b_k > 0$ for $k \ge m$ and $\lim_{k\to\infty} a_k/b_k = L$.

  (i) If $0 < L < \infty$, then $\sum_{k=m}^{\infty} b_k$ and $\sum_{k=m}^{\infty} a_k$ both converge or both diverge.

  (ii) If $L = \infty$ and $\sum_{k=m}^{\infty} b_k$ diverges, then $\sum_{k=m}^{\infty} a_k$ diverges.

  (iii) If $L = 0$ and $\sum_{k=m}^{\infty} b_k$ converges, then $\sum_{k=m}^{\infty} a_k$ converges.

Example 5. Show that $\sum_{k=1}^{\infty} k^{-q}\ln(k)$ converges if $q > 1$ and diverges if $q \le 1$.

Sol. Compare $k^{-q}\ln(k)$ with k^{-p} for some $p \in (1, q)$ if $q > 1$ (limit comparison). Compare $k^{-q}\ln(k)$ with k^{-q} for $q \le 1$ (limit comparison).

- Absolute convergence test. If $\sum_{k=m}^{\infty} |a_k|$ converges, $\sum_{k=m}^{\infty} a_k$ converges. In such case, we say that $\sum_{k=m}^{\infty} a_k$ converges absolutely.

  – Proof of the absolute convergence test is based on the fact that $0 \le |a_k| + a_k \le 2|a_k|$.

  – If $\sum_{k=m}^{\infty} a_k$ converges and $\sum_{k=m}^{\infty} |a_k|$ diverges, then we say that $\sum_{k=m}^{\infty} a_k$ converges conditionally.

Example 6. Show that $\sum_{k=1}^{\infty} 2^{-k}\sin(k)$ converges.

Sol. Compare $|2^{-k}\sin(k)|$ with 2^{-k} (direct comparison) and we have $\sum_{k=1}^{\infty} |2^{-k}\sin(k)|$ converges. By the absolute convergence test, $\sum_{k=1}^{\infty} 2^{-k}\sin(k)$ converges.

- Generalized ratio test. Suppose that $\lim_{k\to\infty} |a_{k+1}/a_k| = L$. Then

  (i) $\sum_{k=m}^{\infty} a_k$ converges absolutely if $L < 1$, and

  (ii) $\sum_{k=m}^{\infty} a_k$ diverges if $L > 1$.

- Alternating series. Suppose that $a_k > 0$ for $k \geq m$ and $\{b_k\} = \{(-1)^k\}$ or $\{(-1)^{k+1}\}$. Then $\sum_{k=m}^{\infty} b_k a_k$ is called an alternating series and the following results holds.

  - Alternating series test. If the sequence $\{a_k\}$ is decreasing and $\lim_{k\to\infty} a_k = 0$, then the alternating series $\sum_{k=m}^{\infty} b_k a_k$ is convergent.
  - Let $S_n = \sum_{k=m}^{n} b_k a_k$. Then $|S_{n+N} - S_n| \leq a_{n+1}$ for $n \geq m$ and $N \geq 0$.

Example 7. Show that $\sum_{k=1}^{\infty}(-1)^{k+1}k^{-1}$ converges conditionally.

Sol. Since $\lim_{k\to\infty} k^{-1} = 0$ and $k^{-1} > 0$, $\sum_{k=1}^{\infty}(-1)^{k+1}k^{-1}$ converges by the alternating series test. Also, $\sum_{k=1}^{\infty}(-1)^{k+1}k^{-1}$ does not converge absolutely since the p-series with $p = 1$ is divergent. Therefore, $\sum_{k=1}^{\infty}(-1)^{k+1}k^{-1}$ converges conditionally.