Limit of a sequence, BMCT and Series

- $\lim_{n\to\infty} a_n$ can be viewed as $\lim_{x\to\infty} f(x)$, where the domain of f is $D = \{n : n \text{ is an integer and } a_n \text{ is defined }\}$, and $f(x) = a_x$ for $x \in D$. We say a sequence $\{a_n\}$ is convergent if $\lim_{n\to\infty} a_n$ exists.

- **Fact 1** *If* $\lim_{x\to\infty} f(x) = L$, *then* $\lim_{n\to\infty} f(n) = L$.

  Example 1. Find $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{n}$.

  Sol. Apply L'Hôpital's rule, $\displaystyle\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$, so $\displaystyle\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$.

- BMCT (bounded, monotonic, convergence theorem): a monotone (單調) sequence is convergent if and only if it is bounded (有界).

- Bounded sequences.

  - A sequence $\{b_n\}$ is bounded above (有上界) if there exists a real number M such that $b_n \le M$ for every n. In such case, M is called an upper bound of $\{b_n\}$.
  - A sequence $\{b_n\}$ is bounded below (有下界) if there exists a real number m such that $b_n \ge m$ for every n. In such case, m is called a lower bound of $\{b_n\}$.
  - A sequence $\{b_n\}$ is bounded if it is bounded both above and below (有界=有上界與下界).

- Monotone (monotonic) sequences (單調數列). A monotone sequence is either an increasing sequence (遞增數列) or a decreasing sequence (遞減數列).

  - A sequence $\{b_n\}$ is increasing (遞增) if $b_n \le b_{n+1}$ for every n. If $b_n < b_{n+1}$ for every n, then $\{b_n\}$ is strictly increasing (嚴格遞增).
  - A sequence $\{b_n\}$ is decreasing (遞減) if $b_n \ge b_{n+1}$ for every n. If $b_n > b_{n+1}$ for every n, then $\{b_n\}$ is strictly decreasing (嚴格遞減).

- The following fact is a modified version of BMCT, and BMCT follows from this fact.

Fact 2 *(Modified BMCT) Suppose that a sequence $\{b_n\}$ is increasing. Then*

(i) $\{b_n\}$ is convergent and converges to its least upper bound (最小上界) if it is bounded above, and

(ii) $\{b_n\}$ diverges to ∞ if it is not bounded above.

- Example 2. Show that the sequence $\{(1+n^{-1})^n\}_{n=1}^{\infty}$ is convergent.

  Sol. Note that

$$\begin{aligned} \frac{(1+(n+1)^{-1})^{n+1}}{(1+n^{-1})^n} &= \left(1-\frac{1}{(n+1)^2}\right)^n\left(1+\frac{1}{n+1}\right) \\ &\geq \left(1-\frac{n}{(n+1)^2}\right)\left(1+\frac{1}{n+1}\right) \\ &= \frac{n^3+3n^2+3n+2}{(n+1)^3} \geq 1 \end{aligned}$$

  and

$$(1+n^{-1})^n \leq \sum_{k=0}^{n}\frac{1}{k!} \leq \sum_{k=0}^{n}\frac{2}{2^k} < 4.$$

  Thus $\{(1+n^{-1})^n\}_{n=1}^{\infty}$ is increasing, bounded above by 4. By the modified BMCT (Fact 2), $\{(1+n^{-1})^n\}$ is convergent.

- Remark. In Example 2, we can also apply the BMCT to show that $\{(1+n^{-1})^n\}_{n=1}^{\infty}$ is convergent since $\{(1+n^{-1})^n\}_{n=1}^{\infty}$ is increasing, bounded above by 4 and bounded below by 2 (the first number in sequence).

- Example 3. Suppose that $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$ for $n \geq 1$. Show that the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent.

- Suppose that $\{a_k\}_{k=m}^{\infty}$ is a sequence of real numbers, where m is fixed. Then

$$\sum_{k=m}^{\infty} a_k \overset{\text{def}}{=} \lim_{n\to\infty}\sum_{k=m}^{n} a_k.$$

  - If $\lim_{n\to\infty}\sum_{k=m}^{n} a_k = L$ exists, then we say the series $\sum_{k=m}^{\infty} a_k$ converges with sum L.
  - If the limit of $\sum_{k=m}^{n} a_k$ does not exist, then we say the series $\sum_{k=m}^{\infty} a_k$ diverges.

- $\sum_{k=m}^{n} a_k$ is called a partial sum ((部分和) of the series $\sum_{k=m}^{\infty} a_k$.

- Examlpes of finding the sum of a series through direct caculation.

  Example 4. Find $\sum_{k=1}^{\infty} 2^{-k}$.

  Sol. Recall that $\sum_{k=1}^{n} ar^k = ar(1-r^n)/(1-r)$ if $r \neq 1$. Therefore,

$$\sum_{k=1}^{\infty} 2^{-k} = \lim_{n\to\infty} \sum_{k=1}^{n} 2^{-k} = \lim_{n\to\infty} \frac{0.5(1-(0.5)^n)}{1-0.5} = 1.$$

  - The geometric series $\sum_{k=m}^{\infty} ar^k = ar^m/(1-r)$ if $|r| < 1$.

  Example 5. Find $\sum_{k=3}^{\infty}(2k+1)/(k^2(k+1)^2)$.

- Suppose that $a_k \geq 0$ for $k \geq m$. From the modified BMCT (Fact 2), we have the following two results.
  - If the partial sum sequence $\{\sum_{k=m}^{n} a_k\}$ is bounded above, then the series $\sum_{k=m}^{\infty} a_k$ converges.
  - If the partial sum sequence $\{\sum_{k=m}^{n} a_k\}$ is not bounded above, then the series $\sum_{k=m}^{\infty} a_k = \infty$ diverges.

- Example 6. Show that the series $\sum_{k=0}^{\infty} 1/k!$ is convergent.

- Divergence test (Theorem 8.9). If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=m}^{\infty} a_k$ diverges.

  Example 7. Suppose that $a \neq 0$. Then the geometric series $\sum_{k=m}^{\infty} ar^k$ diverges if $|r| \geq 1$.