Improper integrals (瑕積分)

- Two basic types of improper integrals:
  (i) integrals with unbounded integration ranges and
  (ii) integrals with discontinuous unbounded integrands.

- Integrals with unbounded integration ranges.
  - Suppose that f is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx. \tag{1}$$

  - Suppose that f is continuous on $(-\infty,b]$, then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx. \tag{2}$$

- Example 1. Find $\displaystyle\int_1^\infty \frac{1}{x^2}dx$.

  Sol.

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}dx &= \lim_{b\to\infty}\left(\int_1^b x^{-2}dx\right)\\ &= \lim_{b\to\infty}\left(\left(-x^{-1}\Big|_1^b\right)\right) = \lim_{b\to\infty}\left(-\frac{1}{b}-(-1)\right) = 1.\end{aligned}$$

- Integrals with unbounded integrands.
  - Suppose that f is continuous on $(a,b]$ and f is unbounded on $[a,b]$. Then

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx. \tag{3}$$

  - Suppose that f is continuous on $[a,b)$ and f is unbounded on $[a,b]$. Then

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx. \tag{4}$$

Example 2. Find $\int_0^{\pi/2}\tan(x)dx$ and $\int_{-\pi/2}^0\tan(x)dx$.

Sol.

$$\begin{aligned}
\int_0^{\pi/2} \tan(x)dx &= \lim_{b\to(\pi/2)^-} \int_0^b \tan(x)dx \\
&= \lim_{b\to(\pi/2)^-} -\ln(\cos(x))\Big|_0^b \\
&= \lim_{b\to(\pi/2)^-} -\ln(\cos(b)) \\
&= \lim_{y\to 0^+} -\ln(y) = \infty.
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi/2}^{0} \tan(x)dx &= \lim_{b\to(-\pi/2)^+} -\ln(\cos(x))\Big|_b^0 \\
&= \lim_{b\to(-\pi/2)^+} \ln(\cos(b)) \\
&= \lim_{y\to 0^+} \ln(y) = -\infty.
\end{aligned}$$

Example 3. Find $\int_0^1 x^{-1/2}dx$.

Sol.

$$\begin{aligned}
\int_0^1 x^{-1/2}dx &= \lim_{b\to 0^+} \int_b^1 x^{-1/2}dx \\
&= \lim_{b\to 0^+} 2x^{1/2}\Big|_b^1 = \lim_{b\to 0^+} (2 - 2b^{1/2}) = 2.
\end{aligned}$$

- Notation. Suppose that $a < b$ and I is an interval with endpoints a and b. Then $\int_I f(x)dx$ means $\int_a^b f(x)dx$.
- Suppose that an interval I can be divided into sub-intervals I_1, …, I_n such that for $k = 1$, …, n, $\int_{I_k} f(x)dx$ can be evaluated as a limit using (1) – (4). Then

$$\int_I f(x)dx = \sum_{k=1}^{n} \int_{I_k} f(x)dx$$

if $\int_{I_1} f(x)dx, \ldots, \int_{I_n} f(x)dx$ can be defined and the sum $\sum_{k=1}^n \int_{I_k} f(x)dx$ does not involving $\infty - \infty$.

  - If $\int_{I_k} f(x)dx$ cannot be defined for some k, then $\int_I f(x)dx$ cannot be defined.

- Limits involving $\pm\infty$ are allowed, but $\infty - \infty$ cannot be defined.
- If $\int_{I_1} f(x)dx, \ldots, \int_{I_n} f(x)dx$ are finite, then the improper integral is called convergent.

- Basic properties of Riemann integrals still hold for improper integrals.
  - Linearity. $\displaystyle\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ and for a constant k, $\displaystyle\int_a^b kf(x)dx = k\int_a^b f(x)dx$.
  - Dominance rule. Suppose that $f \geq g$ on (a, b), then $\displaystyle\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
  - Subdivision rule. $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

- Example 4. Find

  (a) $\displaystyle\int_{-\pi/2}^{\pi/2} \tan(x)dx$ and

  (b) $\displaystyle\int_{-\pi/2}^{\pi/2} |\tan(x)|dx$.

  Sol.

  (a) From Example 2, we have

  $$\int_0^{\pi/2} \tan(x)dx = \infty \text{ and } \int_{-\pi/2}^0 \tan(x)dx = -\infty,$$

  so

  $$\int_{-\pi/2}^{\pi/2} \tan(x)dx = \int_{-\pi/2}^0 \tan(x)dx + \int_0^{\pi/2} \tan(x)dx = -\infty + \infty$$

  cannot be defined.

  (b)

  $$\begin{aligned}\int_{-\pi/2}^{\pi/2} |\tan(x)|dx &= \int_{-\pi/2}^0 (-\tan(x))dx + \int_0^{\pi/2} \tan(x)dx \\ &= -\int_{-\pi/2}^0 \tan(x)dx + \infty \\ &= -(-\infty) + \infty = \infty.\end{aligned}$$

- Example 5. Let

$$f(x) = \begin{cases} 2 & \text{if } |x| \le 1; \\ |x|^{-2} & \text{if } |x| > 1. \end{cases}$$

Find $\displaystyle\int_0^\infty f(x)dx$.

Sol. From Example 1, $\int_1^\infty (1/x^2)dx = 1$, so

$$\int_0^\infty f(x)dx = \int_0^1 2dx + \int_1^\infty \frac{1}{x^2}dx = 2x|_0^1 + 1 = 2 + 1 = 3.$$

- Example 6. Find $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}dx$ using the following results:

  (i) $\displaystyle\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$, and

  (ii) $\lim_{x\to\infty} \tan^{-1}(x) = \pi/2$.

  Answer: π.

- Verification of $\lim_{x\to\infty} \tan^{-1}(x) = \pi/2$. For $\varepsilon \in (0, \pi/2)$, take $M = \tan(\pi/2 - \varepsilon)$, then

$$x > M \Rightarrow \tan^{-1}(x) > \pi/2 - \varepsilon \Rightarrow |\tan^{-1}(x) - \pi/2| < \varepsilon.$$

Thus $\lim_{x\to\infty} \tan^{-1}(x) = \pi/2$. Here we have used the fact that $\tan^{-1}(x)$ is a strictly increasing function of x on $(-\infty, \infty)$ since $\displaystyle\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} > 0$ for $x \in (-\infty, \infty)$.

- Example 7. Find $\int_{-\infty}^{-1} x^{-1}dx$. Answer: $-\infty$.

  Example 8. Find $\int_1^\infty x^{-1}dx$. Answer: ∞.

  Example 9. Let

$$f(x) = \begin{cases} 2 & \text{if } |x| \le 1; \\ 1/x & \text{if } |x| > 1. \end{cases}$$

Find $\int_{-\infty}^\infty f(x)dx$.

Answer: $\int_{-\infty}^\infty f(x)dx$ cannot be defined.