Integration methods

- Integration by substitution (change of variable, 積分換變數) .
  - Change of variable using one-to-one transform.
  - Change of variable when the transform is not one-to-one.

- Change of variable using one-to-one transform. Suppose that we are interested in calculating $\int_a^b T(u)du$ (assuming the functions T is continuous) and would like to do a change of variable by setting $x = g(u)$, where g^{-1} exists. Then we have

$$\begin{aligned} \int_a^b T(u)du &= \int_{g(a)}^{g(b)} \frac{T(u)}{\frac{dx}{du}} dx \\ &= \int_{g(a)}^{g(b)} \left. \frac{T(u)}{g'(u)} \right|_{u=g^{-1}(x)} dx \\ &\overset{T_1(u)=T(u)/g'(u)}{=} \int_{g(a)}^{g(b)} T_1(g^{-1}(x))dx. \end{aligned} \tag{1}$$

- Change of variable when the transform is not one-to-one. Suppose that in (1), $\frac{T(u)}{\frac{dx}{du}}$ can be expressed as a function of x, say $f(x)$, then

$$\int_a^b T(u)du = \int_{g(a)}^{g(b)} \frac{T(u)}{\frac{dx}{du}} dx = \int_{g(a)}^{g(b)} f(x)dx. \tag{2}$$

  In such case, (2) holds even when g is not one-to-one and (2) can be re-written as

$$\int_a^b f(g(u))g'(u)du = \int_{g(a)}^{g(b)} f(x)dx \tag{3}$$

  since $T(u)/g'(u) = f(x) = f(g(u))$.

- Example 1. Find $\int_0^\pi \sin^2(x)\cos(x)dx$.

  Sol.

$$\begin{aligned} \int_0^\pi \sin^2(x)\cos(x)dx &\overset{u=\sin(x)}{=} \int_{\sin(0)}^{\sin(\pi)} \frac{\sin^2(x)\cos(x)}{\frac{du}{dx}} du \\ &= \int_{\sin(0)}^{\sin(\pi)} \frac{\sin^2(x)\cos(x)}{\cos(x)} du = \int_0^0 u^2 du = 0. \end{aligned}$$

  Example 2. Find $\int_0^{2\pi/3} \sin(3x)dx$.

$(u = 3x)$

Example 3. Find $\int_0^1 (1+3x)^5 dx$.

$(u = 1 + 3x)$

- The change of variable formula in (3) can be applied when computing $\int_c^d f(x)dx$ by setting $x = g(u)$. From (3),

$$\int_c^d f(x)dx = \int_a^b f(g(u)) \underbrace{g'(u)}_{dx/du} du,$$

where a and b are constants such that $g(a) = c$ and $g(b) = d$.

- Example 4. Find $\int_0^2 \sqrt{4-x^2}dx$.

Sol. Set $x = 2\cos(\theta)$ and choose $a = \pi/2$, $b = 0$ so that $2\cos(a) = 0$ and $2\cos(b) = 2$ (other choices for (a, b) are also possible). Then

$$\begin{aligned} \int_0^2 \sqrt{4-x^2}dx &= \int_{\pi/2}^0 \sqrt{4-(2\cos(\theta))^2} \frac{d2\cos(\theta)}{d\theta} d\theta \\ &= -\int_0^{\pi/2} |2\sin(\theta)|(-2\sin(\theta))d\theta \\ &= \int_0^{\pi/2} 4\sin^2(\theta)d\theta \\ &= \int_0^{\pi/2} 2(1-\cos(2\theta))d\theta = (2\theta - \sin(2\theta))|_0^{\pi/2} = \pi. \end{aligned}$$

- Notation. $\int_a^b f(x)dg(x)$ means $\displaystyle\int_a^b f(x)\frac{dg(x)}{dx}dx$ and $\int f(x)dg(x)$ means $ds \int f(x)\frac{dg(x)}{dx}dx$.

- Integration by parts (分部積分).

$$\int_a^b u(x)dv(x) = u(x)v(x)|_a^b - \int_a^b v(x)du(x).$$

For indefinite integrals:

$$\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x).$$

  - Recall: we use the indefinite integral $\int f(x)dx$ to denote an antiderivative of f.

Example 5. Find $\int_0^1 xe^x dx$, $\int_0^1 x^2e^x dx$ and $\int xe^x dx$.

Sol.

$$\begin{aligned}\int xe^x dx &= \int x de^x \\ &= xe^x - \int e^x dx = xe^x - e^x. \end{aligned} \tag{4}$$

$$\int_0^1 xe^x dx \overset{(4)}{=} (xe^x - e^x)|_{x=0}^1 = e - e - (0-1) = 1. \tag{5}$$

$$\begin{aligned}\int_0^1 x^2e^x dx &= \int_0^1 x^2 de^x \\ &= x^2e^x|_0^1 - \int_0^1 e^x dx^2 = e - \int_0^1 e^x 2x dx \overset{(5)}{=} e - 2.\end{aligned}$$

Example 6. Show that $\int (\ln x)^n dx = x(\ln x)^n - n\int(\ln x)^{n-1}dx$ for $n \geq 1$.

Sol. Take $u(x) = (\ln x)^n$ and $v(x) = x$ and then apply integration by parts.

Example 7. Show that

$$\int \sec^k(x)dx = \frac{1}{k-1}\left(\tan(x)\sec^{k-2}(x) + (k-2)\int \sec^{k-2}(x)dx\right) \tag{6}$$

for $k \geq 2$.

Sol. Write $\int \sec^k(x)dx = \int \sec^{k-2}(x)d\tan(x)$ and then apply integration by parts.

- Note. When solving the problems in Examples 6 and 7, we use the following properties for indefinite integral.
  (i) For a constant a, $\int af(x)dx = a\int f(x)dx$.
  (ii) $\int(f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$; $\int(f(x)-g(x))dx = \int f(x)dx - \int g(x)dx$.

(iii) Suppose that $\int f(x)dx = a \int f(x)dx + g(x)$ for some constant $a \neq 1$. Then
$$\int f(x)dx = \frac{1}{1-a} g(x).$$

Example 8. Find $\int_0^{\pi/4} \tan^{-1}(x)dx$ using integration by parts.

- Integration of $\sin^m(x)\cos^n(x)$.
  - If m is odd, set $u = \cos(x)$ (substitution). If n is odd, set $u = \sin(x)$.
  - If m and n are even, use $\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$.

Example 9. Find $\int \sin^3(x)\cos^n(x)dx$.

Example 10. Find $\int \sin^2(x)\cos^2(x)dx$.

- Integration of $\tan^m(x)\sec^n(x)$.
  - If m is odd, set $u = \sec(x)$. If n is even, set $u = \tan(x)$.
  - If m is even and n is odd, first express $\tan^2(x)$ using $\sec^2(x)$, and then use the formula in (6) and the fact that
  $$\int \sec(x)dx = \ln|\tan(x) + \sec(x)|$$
  to find $\int \sec^k(x)dx$.
- Integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ and $\sqrt{x^2 - a^2}$.
  - Approach: set $x = a\cos(\theta)$, $a\tan(\theta)$ and $a\sec(\theta)$ respectively.

Example 11. Find $\int_0^1 \sqrt{1 - x^2}dx$.

Example 12. Find $\int_0^1 \sqrt{1 + x^2}dx$.

Example 13. Find $\int_2^3 \sqrt{x^2 - 1}dx$.