Introduction to integration

- Suppose that $f(x) \geq 0$ for x in the closed interval $[a, b]$. Then the integral of f over $[a, b]$ is defined so that it is the area between the curve $y = f(x)$ and the x-axis on $[a, b]$.
- Partition (分割) of an interval $[a, b]$. A partition is a way of dividing an interval into sub-intervals. Suppose that $a = x_0 < x_1 < \cdots < x_n = b$.
  - The interval $[a, b]$ can be divided into n sub-intervals $[x_0, x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, x_n]$. Such a partition of $[a, b]$ is denoted by $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$.
  - The norm (or mesh) of a partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is

$$\max\{x_k - x_{k-1} : 1 \leq k \leq n\}.$$

- Example 1. Consider $\mathcal{P} = \{0, 0.11, 0.32, 0.4, 1\}$ as a partition for $[0, 1]$. Then the norm of $\mathcal{P}$ is

$$\max\{0.11 - 0, 0.32 - 0.11, 0.4 - 0.32, 1 - 0.4\} = 0.6.$$

- Riemann sum (黎曼和). Suppose that f is a real-valued function whose domain contains $[a, b]$ and $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$. Suppose that for $k \in \{1, \ldots, n\}$, $c_k \in [x_{k-1}, x_k]$. Then the sum

$$\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$$

  is called a Riemann sum of f on $[a, b]$. c_1, …, c_n are called the subinterval representatives (子區間代表點) for the Riemann sum.

- Definition of integrals (積分定義). The integral of f on $[a, b]$ is defined as

$$\lim_{\text{norm of } \mathcal{P} \to 0} \underbrace{\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})}_{\text{黎曼和, partition: } \mathcal{P} = \{x_0, x_1, \ldots, x_n\}}, \tag{1}$$

  if the limit exists.

- Suppose that $f(x) = C_0$ is a constant on $[a, b]$, $\sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) = C_0(b - a)$, so $\int_a^b C_0 dx = C_0(b - a)$.

- When the limit in (1) exists, we say that f is integrable (可積) or Riemann integrable (黎曼可積) on $[a, b]$, and the number A is called the Riemann integral (黎曼積分) of f on $[a, b]$, which is denoted by $\int_a^b f(x)dx$.

- Not all functions are integrable. However, if f is continuous on $[a, b]$, then f is integrable on $[a, b]$. In such case, consider the partition that divides the interval $[a, b]$ into n sub-intervals of equal length: $\{x_0, x_1, \ldots, x_n\}$, where

$$x_k = a + \frac{(b-a)k}{n} \text{ for } 0 \leq k \leq n.$$

Then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{(b-a)k}{n}\right).$$

- Example 2. Suppose that $f(x) = x$ for every real number x. Find $\int_0^2 f(x)dx$.

Sol. Since f is continuous on $[0, 2]$, it is integrable on $[0, 2]$ and

$$\begin{aligned}\int_0^2 f(x)dx &= \lim_{n\to\infty} \left(\frac{2}{n}\right) \sum_{k=1}^{n} f\left(\frac{2k}{n}\right) \\ &= \lim_{n\to\infty} \left(\frac{2}{n}\right) \sum_{k=1}^{n} \left(\frac{2k}{n}\right) \\ &= \lim_{n\to\infty} \frac{2n(n+1)}{n^2} = 2.\end{aligned}$$

- Integrals and areas.
  - Suppose that $f(x) \geq 0$ for x in the closed interval $[a, b]$. Then $\int_a^b f(x)dx$ is the area of the region $\{(x, y) : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}$.
  - Suppose that $f(x) \leq 0$ for x in the closed interval $[a, b]$. Then $\int_a^b (-f(x))dx$ is the area of the region $\{(x, y) : a \leq x \leq b \text{ and } f(x) \leq y \leq 0\}$.

- Integrals as averages. If f is integrable on $[a, b]$, then the quantity

$$\frac{\int_a^b f(x)dx}{b-a}$$

can be interpreted as the "average" value of f on $[a, b]$.

- First fundamental theorem of calculus (first FTC, 第一微積分基本定理). Suppose that f is continuous on $[a,b]$ and there exists a function g such that $g' = f$ on $[a,b]$. Then $\int_a^b f(x)dx = g(b) - g(a)$.

  Proof of the first FTC. Let

$$x_k = a + \frac{k(b-a)}{n} \text{ for } 0 \le k \le n,$$

  then

$$g(b) - g(a) = \sum_{k=1}^{n} g(x_k) - g(x_{k-1}) = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) \text{ (by the MVT).}$$

  Let $n \to \infty$, then

$$g(b) - g(a) = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) = \int_a^b f(x)dx.$$

- Suppose that $g' = f$, then g is called an antiderivative (反導函數) of f.

- Example 3. Find $\int_0^2 xdx$.

  Sol. Take $g(x) = \frac{1}{2}x^2$, then $g'(x) = x$, so

$$\int_0^2 xdx = g(2) - g(0) = 2.$$

- We often use the notation $g(x)|_{x=a}^{x=b}$ or $g(x)|_a^b$ to represent $g(b) - g(a)$, so in Example 3, we may write

$$\int_0^2 xdx = \left.\frac{x^2}{2}\right|_0^2 = \frac{2^2}{2} - \frac{0^2}{2}.$$

- Example 4. Let $v(t)$ be the speed (km/hr) of a car at time t (in hours). Suppose that

$$v(t) = \begin{cases} 36000t & \text{if } t \le 1/1200; \\ 30 & \text{if } t \ge 1/1200. \end{cases}$$

  Find the average speed of the car from time 0 to time 1/60.

Sol. Note that v is continuous on $[0, 1/60]$, so it is Riemann integrable. The average speed (in km/hr) is

$$\frac{\int_0^{1/60} v(t)dt}{(1/60) - 0} = 60 \int_0^{1/60} v(t)dt.$$

Let

$$g(t) = \begin{cases} 18000t^2 & \text{if } t \leq 1/1200; \\ 30t - 0.0125 & \text{if } t \geq 1/1200, \end{cases}$$

then $g'(t) = v(t)$ for $t \in (-\infty, \infty)$ (the proof for $g' = v$ is left as an exercise). By the first FTC,

$$\int_0^{1/60} v(t)dt = g(t)|_0^{1/60} = 30(1/60) - 0.0125 - 0 = 0.4875$$

and the average speed from time 0 to time 1/60 is $60(0.4875) = 29.25$ km/hr.

- The expression

$$\int f(x)dx = g(x) + C$$

means that for any constant C, $g + C$ is an antiderivative of f.

  - The notation $\int f(x)dx$ is called the indefinite integral (不定積分) of f.
  - An integral of the form $\int_a^b f(x)dx$ (with given range for integration) is called an definite integral (定積分).

- Indefinite integrals (不定積分) of some basic functions (more can be found in Theorem 5.2).

  - $\int e^x dx = e^x + C$.
  - $\int x^a dx = x^{a+1}/(a+1) + C$ if $a \neq -1$ and $x > 0$.
  - $\int x^{-1} dx = \ln x + C$ for $x > 0$; $\int x^{-1} dx = \ln |x| + C$ for $x < 0$.
  - $\int \sin(x)dx = -\cos(x) + C$.
  - $\int \cos(x)dx = \sin(x) + C$.

- Example 5. Find $\int_0^2 x^3 dx$.

  Sol. Note that

$$\int x^3 dx = \frac{x^4}{4} + C,$$

so we have

$$\int_0^2 x^3 dx = \left.\frac{x^4}{4}\right|_0^2 = 4.$$

- Example 6. Find $\int_{-2}^{-1} x^{-1} dx$.

  Sol.
  $$\int_{-2}^{-1} \frac{1}{x} dx = \ln(-x)|_{-2}^{-1} = \ln(1) - \ln(2) = -\ln(2).$$

- **Fact 1** *Suppose that f and g are real-valued functions defined on $[a, b]$. Suppose that f is integrable on $[a, b]$ and $g(x) = f(x)$ for all but finitely many x's in $[a, b]$. Then g is also integrable on $[a, b]$ and $\int_a^b g(x)dx = \int_a^b f(x)dx$.*

- Example 7. Let

  $$f(x) = \begin{cases} 1 & \text{for } x \neq 2, 4; \\ 0 & \text{if } x = 2 \text{ or } x = 4. \end{cases}$$

  Find $\int_0^5 f(x)dx$.

  Sol. $\int_0^5 f(x)dx = \int_0^5 1dx = 5$.

- Definitions.

  - For $a < b$, $\int_b^a f(x)dx = -\int_a^b f(x)dx$.
  - $\int_a^a f(x)dx = 0$.

- Properties of integrals.

  1. Linearity. Suppose that $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist and k is a constant, then

     $$\int_a^b (f(x) + kg(x))dx = \int_a^b f(x)dx + k \int_a^b g(x)dx.$$

  2. Dominance rule. Suppose that f and g are integrable on $[a, b]$ and $f \geq g$ on $[a, b]$, then

     $$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

3. Subdivision rule. Suppose that $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ exist, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

- Example 8. Find $\int_0^{2\pi}(x+\sin(x))dx$.

  Sol.

$$\begin{aligned}\int_0^{2\pi}(x+\sin(x))dx &= \int_0^{2\pi} xdx + \int_0^{2\pi}\sin(x)dx \text{ (by Linearity)}\\ &= \left.\frac{x^2}{2}\right|_0^{2\pi} + \left(-\cos(x)|_0^{2\pi}\right) = 2\pi^2.\end{aligned}$$

- Example 9. Let

$$f(x) = \begin{cases} 1 & \text{for } x \geq 1; \\ x & \text{if } x < 1. \end{cases}$$

  Find $\int_0^2 f(x)dx$.

  Sol. Note that $f(x) = x$ on $[0,1]$ and $f(x) = 1$ on $[1,2]$, so

$$\int_0^1 f(x)dx = \int_0^1 xdx = \frac{1}{2}$$

  and

$$\int_1^2 f(x)dx = \int_1^2 1dx = 1.$$

  By the subdivision rule,

$$\int_0^2 f(x)dx = \frac{1}{2} + 1 = \frac{3}{2}.$$

- Second fundamental theorem of calculus (second FTC, 第二微積分基本定理): Suppose that f is continuous on an open interval I and $a \in I$. Then

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \text{ for } x \in I. \tag{2}$$

  – If the interval I is not open, then for $x \in I$ and x is not an endpoint of I, (2) still holds. For $x \in I$ and x is the right/left endpoint of I, (2) holds if the derivative is replaced by the left/right derivative.

- The proof of the second FTC is based on the MVT for integration (均值定理積分版):
  - Suppose that f is continuous on $[a, b]$. Then there exists some $c \in [a, b]$ such that $\int_a^b f(x)dx/(b-a) = f(c)$.

- Example 10. Suppose that $f(x) = \int_1^x t^{-1}dt$ for $x \geq 1$. Find $f'(x)$ for $x > 1$.

  Ans. $f'(x) = 1/x$ for $x > 1$.

- Example 11. Suppose that $f(x) = \int_1^{x^2} t^{-1}dt$ for $x \geq 1$. Find $f'(x)$ for $x > 1$.

  Ans. $f'(x) = 2/x$ for $x > 1$.