**L'Hôpital's rule**

- Suppose that $a$ is a real number and $\Delta$ can be $a$, $a^+$, $a^-$, $\infty$ or $-\infty$. Recall that

$$N(\Delta, D) = \begin{cases} (a-D, a) \cup (a, a+D) & \text{if } \Delta = a, D > 0; \\ (a-D, a) & \text{if } \Delta = a^-, D > 0; \\ (a, a+D) & \text{if } \Delta = a^+, D > 0; \\ (D, \infty) & \text{if } \Delta = \infty; \\ (-\infty, D) & \text{if } \Delta = -\infty. \end{cases}$$

- Suppose that $a$ is a real number and $\Delta$ can be $a$, $a^+$, $a^-$, $\infty$ or $-\infty$.
  - We say that $\lim_{x\to\Delta} f(x)/g(x)$ is of the $0/0$ type if
  $$\lim_{x\to\Delta} f(x) = 0 = \lim_{x\to\Delta} g(x).$$
  - We say that $\lim_{x\to\Delta} f(x)/g(x)$ is of the $\infty/\infty$ type if
  $$\lim_{x\to\Delta} f(x) = \infty \text{ and } \lim_{x\to\Delta} g(x) = \infty, \tag{1}$$
  or (1) holds with one (or two) of the $\infty$(s) replaced by $-\infty$(s).

- L'Hôpital's rule (羅比達法則). Suppose that $\lim_{x\to\Delta} f(x)/g(x)$ is of the $0/0$ type or the $\infty/\infty$ type, and
$$\lim_{x\to\Delta} \frac{f'(x)}{g'(x)} = L, \tag{2}$$
where $L$ can be a real number, $\infty$ or $-\infty$. Then
$$\lim_{x\to\Delta} \frac{f(x)}{g(x)} = \lim_{x\to\Delta} \frac{f'(x)}{g'(x)} = L.$$
By writing down (2), it is assumed that on some $N(\Delta, D)$, $f$ and $g$ are differentiable and $g' \neq 0$.

- L'Hôpital's rule for the $0/0$ case with $x \to \Delta$ with $\Delta = a$, $a^+$ or $a^-$ can be proved using a general version of MVT.
  - Generalized MVT. Suppose that $f$ and $g$ are continuous on $[a, b]$ and differentiable on $(a, b)$. Suppose that $g'(x) \neq 0$ for $x \in (a, b)$. Then there exists $c \in (a, b)$ such that
  $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

- Example 1. Find $\lim_{x\to 0}(1-\cos(x))/x$. (Answer: 0)
- Example 2. Find $\lim_{x\to 0}\sin(x)/x$. (Answer: 1)
- Example 3. Find $\lim_{x\to 0}(\sin(x)/x-1)/x$. (Answer: 0)
- Example 4. Find $\lim_{x\to\infty} e^x/x$. (Answer: $\infty$)

  Note: $\lim_{x\to\infty} e^x = \infty$ since for every $M_1 > 0$, $x > \ln(M_1) \Rightarrow e^x > M_1$.
- Example 5. Find $\lim_{x\to 0^+} x^x$. (Answer: 1)
- Example 6. Find $\lim_{x\to\infty}(1+1/x)^x$. (Answer: $e$)
- Example 7. Find $\lim_{x\to\infty}(x+\sin(x))/(x+\cos(x))$. (Answer: 1)
- Proof of l'Hôpital's rule for the $\infty/\infty$ case assuming $L = \lim\limits_{x\to\Delta}\dfrac{f'(x)}{g'(x)}$ is a real number.

  Note that

$$\begin{aligned}
\frac{f(x)}{g(x)} - \frac{f(x)-f(b)}{g(x)-g(b)} &= \frac{-f(x)g(b)+f(b)g(x)}{g(x)(g(x)-g(b))}\\
&= \frac{f(b)(g(x)-g(b))-g(b)(f(x)-f(b))}{g(x)(g(x)-g(b))}\\
&= \frac{f(b)}{g(x)} - \frac{g(b)(f(x)-f(b))}{g(x)(g(x)-g(b))}.
\end{aligned}$$

  Since $\lim\limits_{x\to\Delta}\dfrac{f'(x)}{g'(x)} = L$, for $\varepsilon > 0$, there exists some $D_1$ such that

$$x \in N(\Delta, D_1) \Rightarrow \left|\frac{f'(x)}{g'(x)} - L\right| < \min\left(1, \frac{\varepsilon}{2}\right).$$

  For $b \in N(\Delta, D_1)$, since $\lim_{x\to\Delta} f(x) = \infty$ and $\lim_{x\to\Delta} g(x) = \infty$, we have

$$\lim_{x\to\Delta}\left|\frac{f(b)}{g(x)}\right| + \left|\frac{g(b)}{g(x)}(L+1)\right| = 0, \tag{3}$$

  so there exists $D_2$ such that

$$x \in N(\Delta, D_2) \Rightarrow \left|\frac{f(b)}{g(x)}\right| + \left|\frac{g(b)}{g(x)}(L+1)\right| < \frac{\varepsilon}{2}.$$

Choose $D_3$ such that $N(\Delta, D_3) \subset N(\Delta, D_1) \cap N(\Delta, D_2)$, then for $x \in N(\Delta, D_3)$,

$$\frac{f(x) - f(b)}{g(x) - g(b)} = \frac{f'(c)}{g'(c)}$$

for some $c \in N(\Delta, D_1)$ and

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - L\right| &\leq \left|\frac{f(x)}{g(x)} - \frac{f(x) - f(b)}{g(x) - g(b)}\right| + \left|\frac{f(x) - f(b)}{g(x) - g(b)} - L\right| \\
&= \left|\frac{f(b)}{g(x)} - \frac{g(b)(f(x) - f(b))}{g(x)(g(x) - g(b))}\right| + \left|\frac{f'(c)}{g'(c)} - L\right| \\
&< \left|\frac{f(b)}{g(x)}\right| + \left|\frac{g(b)}{g(x)}\right|(L+1) + \frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore, $\lim_{x\to\Delta} f(x)/g(x) = L$.

- Remarks on the proof of l'Hôpital's rule.
  - For the 0/0 case with finite $L$, we can replace (3) by

    $$\lim_{b\to\Delta} \left|\frac{f(b)}{g(x)}\right| + \left|\frac{g(b)}{g(x)}(L+1)\right| = 0.$$

  - For the case where $L = \infty$, we can use

    $$\frac{f(x}{g(x)} = \frac{f(b)}{f(x)} + \left(1 - \frac{g(b)}{g(x)}\right)\left(\frac{f(x) - f(b)}{g(x) - g(b)}\right).$$

    Since $f(b)/f(x)$ and $g(b)/g(x)$ can be made small, $f(x)/g(x)$ can be made large.
  - The case where $L = -\infty$ can be established by considering $\lim_{x\to\Delta} -\frac{f(x)}{g(x)}$.