Sketching function graphs

- Basic features to be included in the graph of a function f:
  - Rising/falling and critical points
  - Concavity (凹口方向) and inflection points (反曲點)

- Critical number/point. Suppose that $f(c)$ is defined. If $f'(c) = 0$ or f is not differentiable at c, then c is called a critical number and the point $(c, f(c))$ is called a critical point (on the graph of f).

- The graph of f is concave up (凹口向上) on an open interval I means that for every $[a, b] \subset I$, the graph of f is below the line passing $(a, f(a))$ and $(b, f(b))$ on (a, b). That is, for every $[a, b] \subset I$,

$$f(x) < f(a) + \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) \text{ for } x \in (a, b). \tag{1}$$

- The graph of f is concave down (凹口向下) on an open interval I means that for every $[a, b] \subset I$, the graph of f is above the line passing $(a, f(a))$ and $(b, f(b))$ on (a, b).

- Fact 1. Suppose that $f'' > 0$ on an open interval I and $[a, b] \subset I$. Then (1) holds.

  The proof of Fact 1 is given at the end of this handout.

- Concavity. Suppose that $f'' > 0$ on an interval I, then f is concave up on I. If $f'' < 0$ on I, then f is concave down on I.

- Inflection point. Suppose that f is continuous at c. If there exists $\delta > 0$ such that

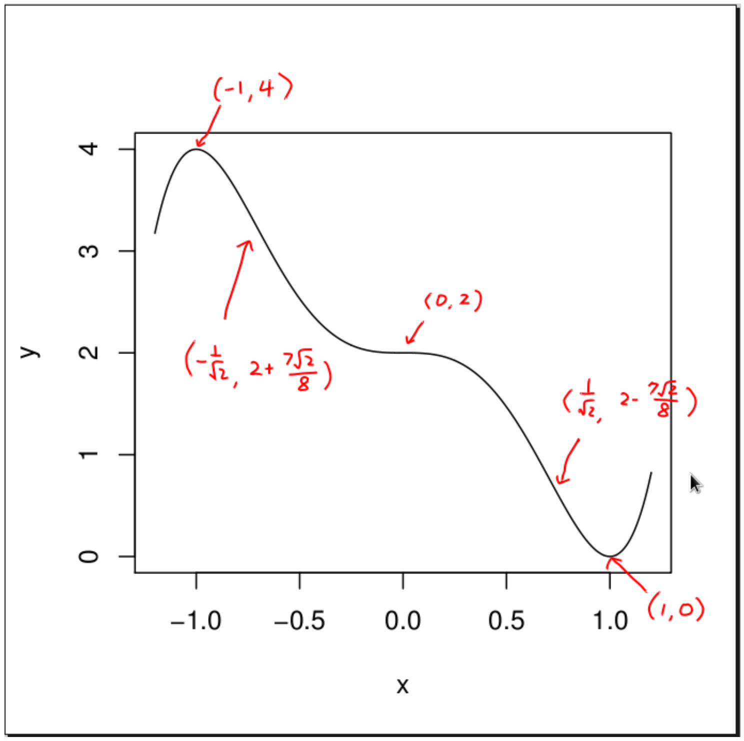
  (i) the graph of f is concave up on $(c, c + \delta)$ and concave down on $(c - \delta, c)$, or

  (ii) the graph of f is concave down on $(c, c + \delta)$ and concave up on $(c - \delta, c)$,

  then the point $(c, f(c))$ is called an inflection point (on the graph of f).

- Example 1. Suppose that $f(x) = 3x^5 - 5x^3 + 2$. Sketch the graph of f and indicate where the graph is rising or falling, where the graph is concave up or concave down, and what the critical point(s) and inflection point(s) are.

  Answer. A sketch of the graph of f is as follows.



  - The graph of f is rising on $(-\infty, -1]$ and $[1, \infty)$ and falling on $[-1, 1]$.
  - The graph of f is concave up on $(-\frac{1}{\sqrt{2}}, 0)$ and $(\frac{1}{\sqrt{2}}, \infty)$ and concave down on $(-\infty, -\frac{1}{\sqrt{2}})$ and $(0, \frac{1}{\sqrt{2}})$.
  - The critical points are $(-1, 4)$, $(0, 2)$ and $(1, 0)$.
  - The inflection points are

$$\left(-\frac{1}{\sqrt{2}}, 2 + \frac{7\sqrt{2}}{8}\right), \quad (0,2) \text{ and } \left(\frac{1}{\sqrt{2}}, 2 - \frac{7\sqrt{2}}{8}\right).$$

- Second derivative test. Suppose that $f'(c) = 0$ and f' exists on some open interval containing c. Then (i) and (ii) hold true.

  (i) If $f''(c) > 0$, then $f(x)$ has a relative minimum at $x = c$.

  (ii) If $f''(c) < 0$, then $f(x)$ has a relative maximum at $x = c$.

  Proof of (i). $f''(c) > 0$ implies that there exists some $\delta > 0$ such that

  $$f'(x) - f'(c) > 0 \text{ if } c < x < c + \delta$$

  and

  $$f'(x) - f'(c) < 0 \text{ if } c - \delta < x < c.$$

  Since f is continuous at c, f has a relative minimum at c.

- Example 2. Suppose that $f(x) = x\sin(x)$. Show that $f(x)$ has a relative minimum at $x = 0$.

- Proof of Fact 1. Let

  $$h(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a),$$

  then

  $$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

  Note that $h'(c) = 0$ for some $c \in (a, b)$ by MVT and that h' is strictly increasing on I since $h'' = f'' > 0$ on I. Therefore, $h'(x) > h'(c) = 0$ for $x \in (c, b)$ and $h'(x) < h'(c) = 0$ for $x \in (a, c)$. Since h is continuous at c, a and b, h is strictly increasing on $[c, b]$ and strictly decreasing on $[a, c]$. Therefore,

  $$h(x) < h(b) = 0 \text{ for } c \leq x < b \text{ and } h(x) < h(a) = 0 \text{ for } a < x \leq c.$$

  Since $h(x) < 0$ for $x \in (a, b)$, (1) holds.