Mean value theorem and its applications

- Rolle's theorem. Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.

  Proof. The extreme value theorem says that there exist c_1 and c_2 in $[a,b]$ such that $f(c_1)$ and $f(c_2)$ are the minimum and maximum of f on $[a,b]$ respectively. Then at least one of the following three cases holds:

  - Case 1. If c_1 is in (a,b), then $f'(c_1) = 0$ and we can take c to be c_1.
  - Case 2. If c_2 is in (a,b), then $f'(c_2) = 0$ and we can take c to be c_2.
  - Case 3. If both c_1 and c_2 are in $\{a,b\}$, then $f(c_1) = f(c_2)$, which implies that f is a constant on $[a,b]$ and c can be any number in (a,b).

  In each of the above cases, we have $f'(c) = 0$ for some $c \in (a,b)$, so Rolle's theorem holds.

- Mean Value Theorem (MVT; 均值定理). Suppose that f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that $f'(c) = (f(b) - f(a))/(b-a)$.

  Proof. Take $h(x) = f(x) - (f(b) - f(a))/(b-a)$ and apply Rolle's theorem.

- Example 1. Find $\displaystyle\lim_{x\to 0^+} \frac{\cos(x) - 1}{x}$ using MVT (mean value theorem).

  Sol. By MVT, for $x > 0$, there exists $c \in (0,x)$ such that

  $$\frac{\cos(x) - 1}{x} = \frac{\cos(x) - \cos(0)}{x - 0} = -\sin(c).$$

  Therefore, for $0 < x < \pi/2$, we have

  $$0 \le \left|\frac{\cos(x) - 1}{x}\right| = \sin(c) \le x.$$

  Since $\lim_{x\to 0^+} x = 0 = \lim_{x\to 0^+} 0$, by the squeeze rule, we have $\lim_{x\to 0^+} |(\cos(x) - 1)/x| = 0$, which implies that $\displaystyle\lim_{x\to 0^+} (\cos(x) - 1)/x = 0$.

- Example 2. Suppose that a car driver drived on a highway for two hours and the driving distance is 400 km. Is it possible that the driver kept the speed under 120 km/hour during the two hours? It is assumed that the driving distance in the first t hours is a differentiable function of t.

  Solution. No, it is not possible. Let $f(t)$ be the driving distance in the first t hours. Then there exists a time point $c \in (0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = 400/2 = 200 > 120.$$

  Therefore, it is not possible that the driving speed was always under 120 km/hour.

- Example 3. Show that $\tan(\theta) \leq 2\theta$ for $0 \leq \theta \leq \pi/4$ by applying MVT.

  Sol. Note that

$$\begin{aligned} \frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)} \\ &= \frac{\left(\frac{d}{dx}\sin(x)\right)\cos(x) - \sin(x)\frac{d}{dx}\cos(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} = \sec^2(x), \end{aligned}$$

  and $1 \leq \sec^2(x) \leq 2$ for $0 \leq x \leq \pi/4$. Apply MVT with $f(x) = \tan(x)$ and $[a, b] = [0, \theta]$, then there exists $c \in [0, \theta]$ such that

$$\frac{\tan(\theta) - \tan(0)}{\theta - 0} = \sec^2(c) \leq 2,$$

  so $\tan(\theta) \leq 2\theta$.

- Zero-derivative theorem. Suppose that f is continuous on $[a, b]$ and $f' = 0$ on (a, b). Then f is a constant on $[a, b]$.

  Proof. MVT.

- Constant difference theorem. Suppose that f and g are continuous on $[a, b]$ and $f' = g'$ on (a, b). Then $f - g$ is a constant on $[a, b]$.

  Proof. Zero-derivative theorem.

Example 4. Suppose that $f'' = 0$ on $(-\infty, \infty)$. Show that $f(x) = a + bx$, where a and b are constants.

- Definitions.
  - f is strictly increasing (嚴格遞增) on an interval (a, b) means that for x_1 and x_2 in (a, b), $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
  - f is increasing (遞增) on an interval (a, b) means that for x_1 and x_2 in (a, b), $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.
  - f is strictly decreasing (嚴格遞減) on (a, b) means that for x_1 and x_2 in (a, b), $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.
  - f is decreasing (遞減) on (a, b) means that for x_1 and x_2 in (a, b), $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

- Monotone function theorem. If $f' > 0$ on (a, b), then f is strictly increasing on (a, b). If $f' < 0$ on (a, b), then f is strictly decreasing on (a, b).

  Proof. MVT.

  Example 5. Suppose that $f(x) = 2x^3 - 9x^2 + 12x + 6$. Find the open interval(s) on which f is strictly increasing or strictly decreasing.

  Answer: f is strictly increasing on $(2, \infty)$ and $(-\infty, 1)$. f is strictly decreasing on $(1, 2)$.

- Fact 1. Suppose that f is differentiable on (a, b). Then (i)-(vi) are true.

  (i) If $f' > 0$ on (a, b) and f is right-continuous at a, then f is strictly increasing on $[a, b)$.

  (ii) If $f' > 0$ on (a, b) and f is left-continuous at b, then f is strictly increasing on $(a, b]$.

  (iii) If $f' > 0$ on (a, b) and f is continuous on $[a, b]$, then f is strictly increasing on $[a, b]$.

  (iv) If $f' < 0$ on (a, b) and f is right-continuous at a, then f is strictly decreasing on $[a, b)$.

  (v) If $f' < 0$ on (a, b) and f is left-continuous at b, then f is strictly decreasing on $(a, b]$.

(vi) If $f' < 0$ on (a, b) and f is continuous on $[a, b]$, then f is strictly decreasing on $[a, b]$.

Proof. MVT. The proof for (ii) is given at the end of this handout.

Example 6. Let $f(x) = e^x - 1 - x$. Find the minimum of f.
Answer: 0.

Example 7. Let $f(x) = x^3 - 1$, then f is strictly increasing on $(-\infty, \infty)$.

- Example 8. Suppose that $f(x) = 3x^5 - 5x^3 + 2$. Find the interval(s) on which f is strictly increasing or strictly decreasing, and determine where f has a relative maximum or a relative minimum.

Sol. $f'(x) = 15x^4 - 15x^2 = 15x^2(x-1)(x+1)$, so

$$f'(x) \begin{cases} > 0 & \text{if } x > 1; \\ < 0 & \text{if } -1 < x < 0 \text{ or } 0 < x < 1; \\ > 0 & \text{if } x < -1, \end{cases}$$

which gives the following results.

(i) By the monotone function theorem, f is strictly increasing on $(-\infty, -1)$ and $(1, \infty)$, and is strictly decreasing on $(-1, 0)$ and $(0, 1)$.

(ii) Since f is continuous at -1, 0 and 1, the results in (i) can be extended as follows: f is strictly increasing on $(-\infty, -1]$ and $[1, \infty)$, and is strictly decreasing on $[-1, 0]$ and $[0, 1]$.

(iii) The two intervals $[-1, 0]$ and $[0, 1]$ in (ii) can be combined since $[-1, 0] \cap [0, 1] \neq \emptyset$. That is, f is strictly decreasing on $[-1, 1]$.

In summary, f is strictly increasing on $(-\infty, -1]$ and $[1, \infty)$, and is strictly decreasing on $[-1, 1]$. Therefore, $f(x)$ has a relative maximum at $x = -1$ and a relative minimum at $x = 1$.

- Proof of Fact 1 (ii).

The proof makes use of Fact 2 and Fact 3, which are stated below.

Fact 2. If f is strictly increasing on (a, b) and $f(b) \geq f(x)$ for $x \in (a, b)$, then f is strictly increasing on $(a, b]$.

Fact 3. Suppose that $\lim_{x\to\Delta} f(x)$ exists and $f(x) > L$ for $x \in N(\Delta, D)$ for some D. Then $\lim_{x\to\Delta} f(x) \geq L$.

We will first show that the result in Fact 1 (ii) holds using Fact 2 and Fact 3, and then prove Fact 2 and Fact 3. Suppose that $f' > 0$ on (a, b) and f is left-continuous on b. Then f is strictly increasing on (a, b) (by monotone function theorem). In addition, for $x_0 \in (a, b)$, $f(x) > f(x_0)$ on (x_0, b), so by Fact 3, $\lim_{x\to b^-} f(x) \geq f(x_0)$ and we have $f(b) = \lim_{x\to b^-} f(x) \geq f(x_0)$ for $x_0 \in (a, b)$. Apply Fact 2, then we have f is strictly increasing on $(a, b]$ and the result in Fact 1 (ii) holds.

It remains to prove Fact 2 and Fact 3.

- Proof of Fact 2. Since

$$f \text{ is strictly increasing on } (a, b), \tag{1}$$

to show that f is strictly increasing on $(a, b]$, it is sufficient to show that

$$f(b) > f(x) \text{ for } x \in (a, b). \tag{2}$$

Since it is assumed that

$$f(b) \geq f(x) \text{ for } x \in (a, b), \tag{3}$$

we only need to show that (3) impies (2) when (1) holds. We will show that (3) impies (2) by contradiction. Suppose (1) and (3) hold and there exists $c \in (a, b)$ such that $f(b) = f(c)$. Then by (1), $f(x) > f(c) = f(b)$ for $x \in (c, b)$, which contradicts with (3). Therefore, (3) impies (2) when (1) holds and the proof of Fact 2 is complete.

- Proof of Fact 3. We will show that $\lim_{x\to\Delta} f(x) \geq L$ by contradiction. Let $L_0 = \lim_{x\to\Delta} f(x)$. Suppose that $L_0 < L$, then there exists D_0 such that

$$x \in N(\Delta, D_0) \Rightarrow |f(x) - L_0| < \frac{L - L_0}{2} \Rightarrow f(x) < L_0 + \frac{L - L_0}{2} < L,$$

so

$$x \in N(\Delta, D_0) \cap N(\Delta, D) \Rightarrow f(x) < L,$$

which contradicts with the assumption that $f(x) > L$ for $x \in N(\Delta, D)$. Therefore, we must have $L_0 \geq L$ and the proof of Fact 3 is complete.