Optimization

- The extreme value theorem (Theorem 4.1, P.185). Suppose that f is continuous on a closed interval $[a, b]$. Then there exists c_1 and c_2 in $[a, b]$ such that

$$f(c_1) \leq f(x) \leq f(c_2) \text{ for every } x \text{ in } [a, b].$$

上述結果說明, 在閉區間$[a, b]$上連續的函數, 會在$[a, b]$上有最大值及最小值.

- 相對極值定義.
  - Local/relative maximum (局部最大值/相對最大值). Suppose that $f(x) \leq f(c)$ for $x \in (c - \delta, c + \delta)$ for some $\delta > 0$, then we say that f has a local/relative maximum at c.
  - Local/relative minimum (局部最小值/相對最小值). Suppose that $f(x) \geq f(c)$ for $x \in (c - \delta, c + \delta)$ for some $\delta > 0$, then we say that f has a local/relative minimum at c.

- Fact 1. Suppose that f is defined on $(c - D, c + D)$ for some $D > 0$ and $f'(c)$ exists. Then the following results hold.

  (a) Suppose that $f'(c) > 0$. Then there exists $\delta > 0$ such that

$$0 < x - c < \delta \Rightarrow f(x) > f(c) \text{ and } -\delta < x - c < 0 \Rightarrow f(x) < f(c).$$

  (b) Suppose that $f'(c) < 0$. Then there exists $\delta > 0$ such that

$$0 < x - c < \delta \Rightarrow f(x) < f(c) \text{ and } -\delta < x - c < 0 \Rightarrow f(x) > f(c).$$

- Revised version of critical number theorem (Theorem 4.2 in Section 4.1). Suppose that f is defined on an open interval (a, b) and $c \in (a, b)$. If f has a relative maximum or a relative minimum at c, then either $f'(c) = 0$ or $f'(c)$ does not exist.
  - Fact 1 implies that if f is differentiable at c and $f'(c) \neq 0$, then f cannot have a local maximum/minimum at c. Therefore, we have the above critical number theorem.
  - The proof of Fact 1 is based on the following claim.

- Claim 1. Suppose that $\lim_{x\to c} g(x) = L \neq 0$ and g is defined on $(c-D, c+D) - \{c\}$ for some $D > 0$. Then there exists $\delta > 0$ such that

$$0 < |x-c| < \delta \Rightarrow |g(x) - L| < \frac{|L|}{2} \Rightarrow L \cdot g(x) > 0.$$

- 由上述 extreme value theorem 及 critical number theorem 可知, 若f 在$[a, b]$上連續且在(a, b)上可微, 則f 在$[a, b]$上有最大值及最小值, 且最大值及最小值只可能發生在a, b 或在集合 $\{c : c \in (a, b), f'(c) = 0\}$ 中的某個點.

- Example 1. Suppose that $f(x) = x^3 - x$. Find the maximum and minimum of f on the interval $[-1, 2]$.

  Sol. $f'(x) = 3x^2 - 1$. Solving $f'(x) = 0$ gives $x = \pm 1/\sqrt{3}$. The maximum and minimum of f are the maximum and minimum of

$$\left\{ f\left(-\frac{1}{\sqrt{3}}\right), f\left(\frac{1}{\sqrt{3}}\right), f(-1), f(2) \right\} = \left\{ \frac{2}{3\sqrt{3}}, -\frac{2}{3\sqrt{3}}, 0, 6 \right\}$$

  respectively. Therefore, the maximum of f is 6 and the minimum of f is $-2/(3\sqrt{3})$.