Derivatives

- Definition 1. The derivative (導數) of a function f at a point c, denoted by $f'(c)$, is

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}. \tag{1}$$

  - If the limit in (1) exists, we say that $f'(c)$ exists, or f is differentiable (可微) at c. $f'(c)$ 叫做f在c的導數, 或f在c的微分.

- 導數的意義

  - If $f(x)$ represents some measurement at time x, then $f'(x)$ represents the instantaneous change rate (瞬間變化率) of the measurement at time x.
  - The slope of the tangent line (切線斜率) to the curve $y=f(x)$ at the point $(c, f(c))$ is $f'(c)$, so one can consider the linear approximation of f:

$$f(x)\approx f(c)+f'(c)(x-c) \text{ when } x\approx c.$$

- Example 1. Suppose that $f(x)=2x$ for $x\in R$. Find $f'(3)$.

  Solution.

$$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{2(3+h)-2(3)}{h}=2.$$

- 不可微的例子.

  Example 2. Suppose that

$$f(x)=\begin{cases} x & \text{if } x\le 0;\\ 2x & \text{if } x>0.\end{cases}$$

  Find $f'(0)$.

  Solution. Note that

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{2(0+h)-2(0)}{h}=2$$

  and

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{0+h-0}{h}=1,$$

so $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ does not exists since the left limit and the right limit are not the same. Therefore, $f'(0)$ does not exist and f is not differentiable at 0.

- Leibniz notation (萊布尼茲符號). $f'(c)$ can be written as $\left.\frac{d}{dx}f(x)\right|_{x=c}$, $\left.\frac{df(x)}{dx}\right|_{x=c}$, or $df(x)/dx|_{x=c}$.

- The notation $\frac{d}{dx}f(x)$ (or $\frac{df(x)}{dx}$, $df(x)/dx$) means $f'(x)$, which is $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

- Example 3. Find $\frac{d}{dx}(2x)$ and $\left.\frac{d}{dx}(2x)\right|_{x=3}$.

  Solution.
  $$\frac{d}{dx}(2x)=\lim_{h\to 0}\frac{2(x+h)-2x}{h}=2,$$
  so $\left.\frac{d}{dx}(2x)\right|_{x=3}=2|_{x=3}=2$.

- f' can be viewed as a function that gives the output value $f'(x)$ for an input value x. The domain of f' is $\{x : f(x)$ is defined and $f'(x)$ exists.$\}$. f' is called the derivative function (導函數) of f.

- If f is a linear function, then f' is a constant function. Specifically, we have $\frac{d}{dx}(a+bx)=b$. The proof of this result is left as an exerice.

- Higher order derivative functions (高階導函數). The n-th derivative function (n階導函數) of f, denoted by $f^{(n)}$, is defined recursively by $f^{(1)}=f'$ and
  $$f^{(n)}(x)=\frac{df^{(n-1)}(x)}{dx} \text{ for } n\geq 2.$$
  - $f^{(n)}(c)$ is called the n-th derivative of f at c (f 在c 的n階函數).
  - $f^{(n)}(x)$ is also written as $\frac{d^n}{dx^n}f(x)$.
  - $f^{(2)}$ is also written as f''.
  - $f^{(3)}$ is also written as f'''.
  - $f^{(0)}$ means f.

- Example 4. $\dfrac{d^2}{dx^2}(2x) = \dfrac{d}{dx}\dfrac{d}{dx}(2x) = \dfrac{d}{dx}2 = 0.$

- Example 5. Show that $\dfrac{d}{dx}\sin(x) = \cos(x)$.

  Sol. Let $f(x) = \sin(x)$ for $x \in R$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1)+\cos(x)\sin(h)}{h} \\ &= \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x). \end{aligned}$$

  Here we have used the fact that $\lim_{x\to 0} \dfrac{1-\cos(x)}{x} = 0$ (the result in Problem 22) and the fact that $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$ (shown in class).

- Remark. One can also show that $\dfrac{d}{dx}\cos(x) = -\sin(x)$ in a similar manner. The proof is left as an exercise.

- Product rule. Suppose both $f'(x)$ and $g'(x)$ exist. Then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x).$$

  Proof. Let $\Delta f = f(x+h) - f(x)$ and $\Delta g = g(x+h) - g(x)$. Then

$$\begin{aligned} \frac{d}{dx}(f(x)g(x)) &= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{(f(x)+\Delta f)(g(x)+\Delta g)-f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x)\Delta g}{h} + \lim_{h\to 0} \frac{g(x)\Delta f}{h} + \lim_{h\to 0} \frac{\Delta f \Delta g}{h} \\ &= f(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + g(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &\quad + \underbrace{\lim_{h\to 0} h \cdot \frac{f(x+h)-f(x)}{h} \cdot \frac{g(x+h)-g(x)}{h}}_{=0\cdot f'(x)\cdot g'(x)=0} \\ &= f(x)g'(x) + g(x)f'(x). \end{aligned}$$

- Example 6. Show that $\frac{d}{dx}x^n = nx^{n-1}$.

  Sol. We will show that $\frac{d}{dx}x^n = nx^{n-1}$ by induction (歸納法). For $n = 1$, we have

$$\frac{d}{dx}x^n = \frac{d}{dx}x = 1 = nx^{n-1}.$$

  Now, suppose that

$$\frac{d}{dx}x^n = nx^{n-1} \tag{2}$$

  for some n. Then

$$\begin{aligned} \frac{d}{dx}(x^{n+1}) &= \frac{d}{dx}(x^n \cdot x) \\ &= \left(\frac{d}{dx}x^n\right)x + \left(\frac{d}{dx}x\right)x^n \\ &= nx^{n-1} \cdot x + 1 \cdot x^n = (n+1)x^n, \end{aligned}$$

  so (2) holds with n replaced by $n+1$. From the above results and induction, (2) holds for every positive integer n.

- Linearity. Suppose that both $f'(x)$ and $g'(x)$ exist and k is a constant. Then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

  and

$$\frac{d}{dx}(kf(x)) = kf'(x).$$

- Example 7. Find $\frac{d}{dx}(x^2 - 2x^3)$ and find the slope of the tangent line to the graph of $y = x^2 - 2x^3$ at the point $(1, -1)$.

  Sol.

$$\frac{d}{dx}(x^2 - 2x^3) = \frac{d}{dx}(x^2) - 2\frac{d}{dx}(x^3) = 2x - 2(3x^2) = 2x - 6x^2.$$

  The slope of the tangent line to the graph of $y = x^2 - 2x^3$ at the point $(1, -1)$ is

$$\left.\frac{d}{dx}(x^2 - 2x^3)\right|_{x=1} = (2x - 6x^2)|_{x=1} = 2 - 6 = -4.$$

- Quotient rule. Suppose both $f'(x)$ and $g'(x)$ exist and $g(x) \neq 0$. Then

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}.$$

The proof of the quotient rule is left as an exercise.

Example 8. Find $\dfrac{d}{dx}\dfrac{\sin(x)}{x}$.

Ans. $\dfrac{x\cos(x) - \sin(x)}{x^2}$.

- Chain rule (連鎖律). Suppose that both $g'(x)$ and $f'(g(x))$ exist. Then

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

Use Leibniz notation, chain rule is expressed as:

$$\frac{d}{dx}f(g(x)) = \frac{d}{dy}f(y)\bigg|_{y=g(x)} \frac{d}{dx}g(x).$$

The proof of the chain rule is based on the fact that if f is differentiable at c, then there exists a function ε that is continuous at c and $\varepsilon(c) = 0$ such that

$$f(c+h) = f(c) + f'(c)h + \varepsilon(c+h)h.$$

The proof of this fact is left as an exercise.

- Example 9. Find $\dfrac{d}{dx}(x^2+1)^{100}$.

  Ans. $200x(x^2+1)^{99}$.

- Derivatives of exponential and logarithmic functions.
  - Define $e = \lim_{n\to\infty}\left(1 + \dfrac{1}{n}\right)^n$.
  - $\log_e(x)$ is often written as $\log(x)$ or $\ln(x)$.
  - $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$ (Theorem 3.9).
  - $\dfrac{d}{dx}e^x = e^x$ (Theorem 3.8).

- Note that in the text, the proof for Theorem 3.8 is informal and the proof of Thoerem 3.9 only deals with $\lim_{h\to 0^+}(\ln(x+h)-\ln(x))/h$. Both theorems can be established using the following results.

  (a) e^x and $\ln(x)$ are continuous.

  (b) $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

  (c) $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$.

  The limit in (c) can be derived using the limit in (b), and the proof for (b) $e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ is based on the fact that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists and e is defined as the limit, and

  $$\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} \le \left(1+\frac{1}{x}\right)^x \le \left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1},$$

  where $\lfloor x\rfloor$ is the integer such that $\lfloor x\rfloor \le x < \lfloor x\rfloor + 1$.

- Example 10. Find $d2^x/dx$.

  Ans. $\ln(2)\cdot 2^x$

- Example 11. Find $d\log_2(x)/dx$.

  Ans. $1/(x\ln(2))$.

- Example 12. Suppose that a is a real number and $f(x) = x^a$ for $x > 0$. Find $f'(x)$.

  Ans. ax^{a-1}.

- Implicit differentiation (隱函數微分). Suppose that $y = f(x)$ is a differentiable function satisfying $g_1(x,y) = g_2(x,y)$. Then in many cases, $f'(x)$ can be found by solving

  $$\frac{d}{dx}g_1(x,f(x)) = \frac{d}{dx}g_2(x,f(x)).$$

- Example 13. Suppose that $y = f(x) > 0$ and $x^2 + y^2 = 25$. Find $f'(3)$ and $f''(3)$.

Sol. Taking derivatives with respect to x at both sides of $x^2 + y^2 = 25$ and we have

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) = 0 \quad &\Rightarrow \quad 2x + \left(\frac{d}{dy} y^2 \Big|_{y=f(x)} \right) f'(x) = 0 \\ &\Rightarrow \quad 2x + 2f(x)f'(x) = 0. \qquad (3) \end{aligned}$$

When $x = 3$, from $3^2 + (f(3))^2 = 25$ and $f(3) > 0$, we have $f(3) = 4$. Plug in $x = 3$ and $f(3) = 4$ in (3) and we have

$$2 \cdot 3 + 2 \cdot 4 \cdot f'(3) = 0,$$

so $f'(3) = -3/4$.

To find $f''(3)$, take derivatives with respect to x at both sides of Equation (3). Then

$$2 + 2(f'(x)f'(x) + f(x)f''(x)) = 0.$$

Plug in $x = 3$, $f'(3) = -3/4$ and $f(3) = 4$ in the above equation and we have

$$2 + 2(9/16 + 4f''(3)) = 0,$$

which gives $f''(3) = -25/64$.

- Example 14. Find the slope of the tangent line to the graph of $x^2 + y^2 = 25$ at the point $(3, 4)$.

  Ans. $-3/4$.

- Example 15. Find the slope of the tangent line to the graph of $x^2y + 2y^3 = 3 + 2y$ at the point $(\sqrt{3}, 1)$.

- Example 16. Suppose that $y = \sin^{-1}(x)$. Find dy/dx.