Limits involving infinity

- Definition 1. Suppose that $L \in R$. $\lim_{x\to\infty} f(x) = L$ means that

$$\underbrace{\text{for every } \varepsilon > 0,}_{\text{給定誤差限制 } \varepsilon} \quad \underbrace{\text{there exists } D}_{\text{可找到控制參數}D}$$

such that (for x in the domain of f)

$$\underbrace{x > D}_{\text{當}x\text{控制在 }(D,\infty)\text{ 範圍}} \qquad \Rightarrow \qquad \underbrace{|f(x) - L| < \varepsilon}_{\text{誤差}f(x)-L\text{就可以控制在}\pm\varepsilon\text{範圍}} \qquad .$$

- Definition 2. Suppose that $L \in R$. $\lim_{x\to-\infty} f(x) = L$ means that

$$\text{for every } \varepsilon > 0, \quad \underbrace{\text{there exists } D}_{\text{可找到控制參數}D}$$

such that (for x in the domain of f)

$$\underbrace{x < D}_{\text{當}x\text{控制在 }(-\infty,D)\text{ 範圍}} \qquad \Rightarrow |f(x) - L| < \varepsilon.$$

- Example 1. Show that $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$.

  Sol. For $\varepsilon > 0$, take $D = 1/\varepsilon$, then

$$x > D \Rightarrow \left|\frac{1}{x} - 0\right| = \frac{1}{x} < \frac{1}{D} = \varepsilon.$$

  Therefore, $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$.

- Example 2. Below is the proof for $\lim_{x\to\Delta} 2^x = L$. Determine what Δ and L are.

  For $\varepsilon > 0$, take $D = \log_2 \varepsilon$, then

$$x < D \Rightarrow |2^x - 0| = 2^x < 2^D = \varepsilon.$$

  Ans. $\Delta = -\infty$ and $L = 0$.

- Remark. Suppose that $L \in R$ and Δ can be a, a^+, a^-, ∞ or $-\infty$. $\lim_{x\to\Delta} f(x) = L$ means that

$$\text{for every } \varepsilon > 0, \quad \underbrace{\text{there exists } D}_{\text{可找到控制參數}D}$$

such that (for x in the domain of f)

$$x \in N(\Delta, D) \Rightarrow |f(x) - L| < \varepsilon,$$

where

$$N(\Delta, D) = \begin{cases} (a-D, a+D) - \{a\} & \text{if } \Delta = a; \\ (a-D, a) & \text{if } \Delta = a^-; \\ (a, a+D) & \text{if } \Delta = a^+; \\ (D, \infty) & \text{if } \Delta = \infty; \\ (-\infty, D) & \text{if } \Delta = -\infty. \end{cases} \tag{1}$$

- $\lim_{x\to\Delta} f(x) = \infty$.

Definition 3. Suppose that Δ can be a, a^+, a^-, ∞ or $-\infty$. $\lim_{x\to\Delta} f(x) = \infty$ means that

$$\underbrace{\text{for every } M}_{\text{給定門檻限制 } M}, \text{ there exists } D$$

such that (for x in the domain of f)

$$x \in N(\Delta, D) \Rightarrow \underbrace{f(x) > M}_{f(x)\text{就可以超過門檻}M}.$$

注意: Definition 3 中, 可假設 $M >$某一個給定的值.

- $\lim_{x\to\Delta} f(x) = -\infty$.

Definition 4. Suppose that Δ can be a, a^+, a^-, ∞ or $-\infty$. $\lim_{x\to\Delta} f(x) = -\infty$ means that

$$\underbrace{\text{for every } M}_{\text{給定門檻限制 } M}, \text{ there exists } D$$

such that (for x in the domain of f)

$$x \in N(\Delta, D) \Rightarrow \underbrace{f(x) < M}_{f(x)\text{就可以低於門檻}M}.$$

注意: Definition 4 中, 可假設 $M <$某一個給定的值.

- Example 3. Show that $\lim_{x\to 1}\dfrac{1}{|x-1|}=\infty$.

  Sol. For $M>0$, take $\delta = 1/M$, then

  $$0<|x-1|<\delta \Rightarrow \frac{1}{|x-1|}>M.$$

  Therefore, $\lim_{x\to 1}\dfrac{1}{|x-1|}=\infty$.

- Example 4. Show that $\lim_{x\to\infty} 2x=\infty$.

  Sol. For a real number M, take $D=M/2$, then

  $$x>D\Rightarrow 2x>M.$$

  Therefore, $\lim_{x\to\infty} 2x=\infty$.

- When $\lim_{x\to\Delta} f(x)=\infty$ (or $-\infty$), we say that $\lim_{x\to\Delta} f(x)$ diverges to ∞ (or $-\infty$) (極限發散到∞ 或$-\infty$).

- 注意: 極限發散到∞ 或$-\infty$, 算是極限不存在的情況. 但求$\lim_{x\to\Delta} f(x)$時, 若能確定是發散到∞ 或$-\infty$, 答案都會寫 $\lim_{x\to\Delta} f(x)=\infty$ (或 $-\infty$), 而不會只說極限不存在. 例如問題是 "Find $\lim_{x\to\infty} 2x$", 答案就會寫 $\lim_{x\to\infty} 2x=\infty$.

- Example 5. Below is the proof for $\lim_{x\to\Delta}\dfrac{1}{x}=L$. Determine what Δ and L are.

  For $M>0$, take $\delta=1/M$, then

  $$0<x<\delta\Rightarrow \frac{1}{x}>\frac{1}{\delta}=M.$$

  Ans. $\Delta=0^+$ and $L=\infty$.

- Example 6. Below is the proof for $\lim_{x\to\Delta}\dfrac{1}{x}=L$. Determine what Δ and L are.

  For $M<0$, take $\delta=-1/M$, then

  $$-\delta<x<0\Rightarrow 0<-x<\delta\Rightarrow \frac{1}{-x}>\frac{1}{\delta}\Rightarrow \frac{1}{x}=-\frac{1}{-x}<-\frac{1}{\delta}=M.$$

  Ans. $\Delta=0^-$ and $L=-\infty$.

- Limit of x rule and constant rule
  - $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to-\infty} x = -\infty$.
  - $\lim_{x\to\Delta} C = C$ for any constant C.
- 牽涉$\pm\infty$ 的極限四則運算規則. 定義

$$\pm\infty \pm \text{實數} = \pm\infty;$$

$$\pm\infty \cdot \text{正實數} = \pm\infty;$$

$$\pm\infty \cdot \text{負實數} = \mp\infty;$$

$$\frac{\text{實數}}{\pm\infty} = 0;$$

$$\infty \cdot \infty = \infty = (-\infty)\cdot(-\infty);$$

$$\infty \cdot (-\infty) = -\infty,$$

  則極限四則運算仍然可以成立.

- $\infty - \infty$, $\pm\infty/\infty$, $0 \cdot \pm\infty$ 無法定義.
- Example 7. Find $\lim_{x\to\infty}(x-1)$.

  Sol. $\lim_{x\to\infty}(x-1) = \lim_{x\to\infty} x - \lim_{x\to\infty} 1 = \infty - 1 = \infty$.
- Example 8. Find $\lim_{x\to-\infty} 1/x$.

  Sol. $\displaystyle\lim_{x\to-\infty} \frac{1}{x} = \frac{1}{-\infty} = 0$.
- Example 9. Find $\lim_{x\to-\infty} x^2$.

  Sol. $\lim_{x\to-\infty} x^2 = (\lim_{x\to-\infty} x)\cdot(\lim_{x\to-\infty} x) = \infty\cdot\infty = \infty$.
- Example 10. Find $\displaystyle\lim_{x\to\infty} \frac{x}{x+1}$.

  Sol. $\displaystyle\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{1}{x+1} = 1 - \frac{1}{\infty} = 1 - 0 = 1$.
- **Fact 1** *Suppose that Δ can be a, a^+, a^- , ∞ or $-\infty$ and L can be a real number, ∞, or $-\infty$. Suppose that there exists D such that*

$$f(x) = g(x) \text{ for every } x \text{ in } N(\Delta, D),$$

  where $N(\Delta, D)$ is defined in (1), and $\lim_{x\to\Delta} g(x) = L$. Then

$$\lim_{x\to\Delta} f(x) = \lim_{x\to\Delta} g(x) = L.$$

Example 11. Suppose that $f(x) = x$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

Sol. $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 0 = 0$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x = \infty$.

- Change of variable. Suppose that Δ can be a, a^+, a^-, ∞, or $-\infty$ and L can be a real number, ∞, or $-\infty$. Suppose that $\lim_{x\to\Delta} g(x) = L$.
  - Suppose that $g(x) \neq L$ for $x \in N(\Delta, D)$ for some D, then
  $$\lim_{x\to\Delta} f(g(x)) = \lim_{y\to L} f(y)$$
  if $\lim_{y\to L} f(y)$ is defined.
  - Suppose that $L \in R$ and $g(x) > L$ for x in $N(\Delta, D)$ for some D, then
  $$\lim_{x\to\Delta} f(g(x)) = \lim_{y\to L^+} f(y)$$
  if $\lim_{y\to L^+} f(y)$ is defined.
  - Suppose that $L \in R$ and $g(x) < L$ for x in $N(\Delta, D)$ for some D, then
  $$\lim_{x\to\Delta} f(g(x)) = \lim_{y\to L^-} f(y)$$
  if $\lim_{y\to L^-} f(y)$ is defined.

- Example 12. Find $\lim_{x\to 0^-}(1/x)$.

  Sol. Let $y = -x$, then $\lim_{x\to 0^-} y = 0$ and $y > 0$ for $x \in N(0^-, D) = (-D, 0)$ for $D > 0$. Therefore, $\lim_{x\to 0^-}(1/x) = \lim_{y\to 0^+} 1/(-y) = -\infty$.

- Example 13. Suppose that $\lim_{x\to\infty} f(-x) = L$. Find $\lim_{x\to-\infty} f(x)$.

  Sol. Let $y = -x$, then $\lim_{x\to\infty} y = -\infty$ and $y \neq -\infty$ for $x \in N(\infty, D) = (D, \infty)$ for D. Therefore, $\lim_{x\to-\infty} f(x) = \lim_{y\to\infty} f(-y) = L$.

- Example 14. Suppose that $\lim_{x\to\infty} f(x) = -\infty$. Find $\lim_{x\to 0^+} f(1/x)$.

  Sol. Let $y = 1/x$, then $\lim_{x\to 0^+} y = \infty$ and $y \neq \infty$ for $x \in N(0^+, D) = (0, D)$ for $D > 0$. Therefore, $\lim_{x\to 0^+} f(1/x) = \lim_{y\to\infty} f(y) = -\infty$.

- One-sided limit theorem: suppose that a is a real number and L can be a real number, ∞, or $-\infty$. Then

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$$

- Example 15. Find $\lim_{x\to 0} \frac{1}{|x|}$ using the fact that $\lim_{x\to 0^+} \frac{1}{x} = \infty$.

  Sol. Since
  $$\lim_{x\to 0^+} \frac{1}{|x|} = \lim_{x\to 0^+} \frac{1}{x} = \infty$$
  and
  $$\lim_{x\to 0^-} \frac{1}{|x|} = \lim_{x\to 0^-} -\frac{1}{x} = \lim_{y\to 0^+} \frac{1}{y} = \infty,$$
  by one-sided limit theorem, $\lim_{x\to 0} \frac{1}{|x|} = \infty$.

- Example 16. Find $\lim_{x\to 0} \frac{1}{x}$.

  Sol. From one-sided limit theorem, $\lim_{x\to 0} \frac{1}{x}$ cannot be a real number since $\lim_{x\to 0^+} \frac{1}{x} = \infty$. Also, $\lim_{x\to 0} \frac{1}{x}$ cannot be ∞ since $\lim_{x\to 0^-} \frac{1}{x} = -\infty$. The limit $\lim_{x\to 0} \frac{1}{x}$ does not exist and is not ∞ or $-\infty$.

- Squeeze rule (夾擠定理). Suppose that Δ can be a, a^+, a^-, ∞, or $-\infty$. Suppose that $g(x) \le f(x) \le h(x)$ for every $x \in N(\Delta, D)$ for some D, then

  $$\lim_{x\to\Delta} g(x) = \lim_{x\to\Delta} h(x) \Rightarrow \lim_{x\to\Delta} f(x) = \lim_{x\to\Delta} g(x) = \lim_{x\to\Delta} h(x).$$

  In addition, we have the following.

  - Suppose that $g(x) \le f(x)$ for every $x \in N(\Delta, D)$ for some D and $\lim_{x\to\Delta} g(x) = \infty$, then $\lim_{x\to\Delta} f(x) = \infty$.
  - Suppose that $f(x) \le g(x)$ for every $x \in N(\Delta, D)$ for some D and $\lim_{x\to\Delta} g(x) = -\infty$, then $\lim_{x\to\Delta} f(x) = -\infty$

- Example 17. Find $\lim_{x\to\infty} x^{2.5}$.

  Sol. Note that $x^{2.5} \ge x^2$ for $x \in N(\infty, 1) = (1, \infty)$ and $\lim_{x\to\infty} x^2$. Therefore, $\lim_{x\to\infty} x^{2.5} = \infty$.

- Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers. The definition of $\lim_{n\to\infty} a_n = L$ is the same as $\lim_{x\to\infty} f(x) = L$ by replacing x by n and $f(x)$ by a_n:
  - Definition for $\lim_{n\to\infty} a_n = L$, $L \in R$.

    $$\underbrace{\text{for every } \varepsilon > 0}_{\text{給定誤差限制 } \varepsilon},\ \underbrace{\text{there exists } D}_{\text{可找到控制參數} D}$$

    such that

    $$n > D \Rightarrow |a_n - L| < \varepsilon.$$

  - Definition for $\lim_{n\to\infty} a_n = \infty$.

    $$\text{for every } M,\ \text{there exists } D$$

    such that

    $$n > D \Rightarrow a_n > M.$$

  - Definition for $\lim_{n\to\infty} a_n = -\infty$.

    $$\text{for every } M,\ \text{there exists } D$$

    such that

    $$n > D \Rightarrow a_n < M.$$

- Suppose that $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$.

Example 18. $\lim_{n\to\infty} \frac{1}{n} = 0$.