Continuity

- Definition 1. f is continuous at a (f 在a點連續) means that

$$\lim_{x\to a} f(x) = f(a).$$

Definition 2. f is right-continuous at a (f 在a點右連續) means that

$$\lim_{x\to a^+} f(x) = f(a).$$

Definition 3. f is left-continuous at a (f 在a點左連續) means that

$$\lim_{x\to a^-} f(x) = f(a).$$

- Suppose that I is an open interval. f is continuous on I means that f is continuous at every point in I.
  - Polynomials are continuous on $R = (-\infty, \infty)$.
  - The absolute value function is continuous on R.
  - sin and cos are continuous on R.

- Example 1. Suppose that $B > 1$. Let $f(x) = B^x$ for $x \in R$. Show that f is continuous on R.

Sol. We need to show that

$$\lim_{x\to a} B^x = B^a \text{ for any } a \in R. \tag{1}$$

For $B^a > \varepsilon > 0$, take

$$\delta = \min\left(\log_B(B^a+\varepsilon) - a, a - \log_B(B^a - \varepsilon)\right),$$

then

$$x \in (a-\delta, a+\delta) - \{a\} \Rightarrow x \in (\log_B(B^a-\varepsilon), \log_B(B^a+\varepsilon)) \Rightarrow |B^x - B^a| < \varepsilon.$$

Therefore, (1) holds and f is continuous on R.

- Remark. From Example 1, for $0 < B < 1$ and $a \in R$, $\lim_{x\to a}(1/B)^x = (1/B)^a$, which implies

$$\lim_{x\to a} B^x = \lim_{x\to a} \frac{1}{(1/B)^x} = \frac{\lim_{x\to a} 1}{\lim_{x\to a}(1/B)^x} = \frac{1}{(1/B)^a} = B^a.$$

Therefore, (1) holds for $0 < B < 1$. In addition, it is clear that (1) holds for $B = 1$, so (1) holds for $B > 0$. This implies that any exponential function is continuous on R.

- The fact that logarithmic functions are continuous on $(0, \infty)$ can be established using the approach in Example 1.

- Example 2. Suppose that n is a positive integer. Let $f(x) = x^{1/n}$ for $x \geq 0$. Show that f is continuous on $(0, \infty)$.

  Sol. Recall that in Example 9 in the handout "Definitions and properties for limits", we have

$$\lim_{x \to a} g(x) = 1 \Rightarrow \lim_{x \to a} (g(x))^{1/n} = 1. \tag{2}$$

  From (2), we can show that

$$\lim_{x \to a} h(x) = L > 0 \Rightarrow \lim_{x \to a} (h(x))^{1/n} = L^{1/n} \tag{3}$$

  by taking $g(x) = h(x)/L$ and apply (2). Apply (3) with $h(x) = x$, then we have

$$\lim_{x \to a} x^{1/n} = a^{1/n} \text{ for } a > 0 \tag{4}$$

  since $\lim_{x \to a} x = a > 0$. (4) means that f is continuous on $(0, \infty)$.

- Example 3. Suppose that n is a positive integer. Let $f(x) = x^{1/n}$ for $x \geq 0$. Show that f is right-continuous at 0.

  Sol. For $\varepsilon > 0$, take $\delta = \varepsilon^n$. Then

$$x \in N(0^+, \delta) = (0, \delta) \Rightarrow 0 < x < \varepsilon^n \Rightarrow |f(x) - 0| = x^{1/n} < \varepsilon.$$

  Therefore, $\lim_{x \to 0^+} f(x) = 0 = f(0)$ and f is right-continuous at 0.

- Continuity on an interval that is not open.

  - f is continuous on $[a, b]$ means that f is continuous on (a, b) and f is right-continuous at a and left-continuous at b. Here both a and b are real numbers.
  - f is continous on $[a, L)$ means that f is continuous on (a, L) and f is right-continous at a. Here a is a real number and L can be a real number or ∞.
  - f is continous on $(L, b]$ means that f is continuous on $(L, b]$ and f is left-continous at b. Here b is a real number and L can be a real number or $-\infty$.

- The f in Examples 2 and 3 is continuous on $[0, \infty)$.

- Removable discontinuity (可修正的不連續). Suppose that $\lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$ for some real number L. If $f(a) \neq L$ or $f(a)$ is not defined, then f is not continuous at a, yet the discontinuity at a is removable in the sense that one can define

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a; \\ L & \text{if } x = a, \end{cases}$$

then g can be viewed as a modified version of f and g is continuous at a.

- Example 4. The function $\dfrac{\sin(x)}{x}$ has a removable discontinuity at 0.

- Composition limit rule: suppose that a is a real number and Δ can be a, a^+, a^-, ∞ or $-\infty$. Suppose that $\lim_{x\to\Delta} g(x)$ exists and f is continuous at $\lim_{x\to\Delta} g(x)$, then $\lim_{x\to\Delta} f(g(x)) = f(\lim_{x\to\Delta} g(x))$.

- Proof of Composition limit rule. Let $L = \lim_{x\to\Delta} g(x)$, then $L \in R$. Since $\lim_{y\to L} f(y) = f(L)$, for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$y \in N(L, \delta) = (L-\delta, L+\delta) - \{L\} \Rightarrow |f(y) - f(L)| < \varepsilon,$$

which implies

$$|y - L| < \delta \Rightarrow |f(y) - f(L)| < \varepsilon. \tag{5}$$

Since $L = \lim_{x\to\Delta} g(x)$, for the δ in (5), there exists D such that

$$x \in N(\Delta, D) \Rightarrow |g(x) - L| < \delta \overset{(5)}{\Rightarrow} |f(g(x)) - f(L)| < \varepsilon.$$

Therefore, $\lim_{x\to\Delta} f(g(x)) = f(L) = f(\lim_{x\to\Delta} g(x))$.

- Example 5. Find $\lim_{x\to 1} \sin(x^3 + 3x + 1)$.

  Sol. Since sin is continuous on R, $\lim_{x\to 1} \sin(x^3+3x+1) = \sin(\lim_{x\to 1}(x^3+3x+1)) = \sin(5)$.

- Example 6. Suppose that $f(x) = \sin(x)/x$ for $x \neq 0$ and f is continuous at 0. Suppose that $g(x) = x$ for $x > 0$ and $g(x) = 0$ for $x \leq 0$. Find $\lim_{x\to 0} f(g(x))$.

  Sol. Since f is continuous on R, $\lim_{x\to 0} f(g(x)) = f(\lim_{x\to 0} g(x)) = f(0) = \lim_{x\to 0} \sin(x)/x = 1$.

- Properties of continuous functions.

(i) Suppose that f and g are continuous at a and c is a constant. Then cf , $f+g$, $f-g$ and $f \cdot g$ are continuous at a. f/g is continuous at a if $g(a) \neq 0$.

(ii) Suppose that g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Note. The properties in (i) still hold true if the continuity at a is replaced by the left or right continuity at a.

- Intermediate value theorem (中間值定理. Theorem 2.6 in the text). Suppose that f is continuous on the interval $[a, b]$ and L is a number between $f(a)$ and $f(b)$ such that $L \neq f(a)$ and $L \neq f(b)$. Then there exists some c in (a, b) such that $f(c) = L$.

  – Special case: root location theorem (勘根定理). Suppose that f is continuous on the interval $[a, b]$ and $f(a)f(b) < 0$. Then there exists some c in (a, b) such that $f(c) = 0$.

- Example 7. Suppose that $f(x) = x(x-1)(x-2) + 0.125$. Show that f has a root in $(0, 1.5)$, a root in $(1.5, 2)$ and a negative root.

  Sol. Note that f is continuous on R, so by root location theorem, $f(a)f(b) > 0$ implies that f has a root between a and b. Since $f(0) > 0$, $f(1.5) < 0$ and $f(2) > 0$, f has at least one root in $(0, 1.5)$ and at least one root in $(1.5, 2)$. Also, $\lim_{x\to-\infty} f(x) = -\infty$, so there exists D such that

$$x \in N(-\infty, D) = (-\infty, D) \Rightarrow f(x) < 0.$$

  Since $f(0) > 0$, we have $D \leq 0$ and $f(D-1) < 0$, so f has at least one root in $(D-1, 0)$, which is a negative root.

- Example 8. Let

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \text{ and } 1/x \text{ is an integer;} \\ 0 & \text{otherwise.} \end{cases}$$

  Then f is continuous at 0 but the graph of f has many holes near 0.