Definitions and properties for limits

- Let
$$f(x)=\frac{2\sqrt{1+x}-x-2}{x^2}.$$
Can we find $\lim_{x\to 0} f(x)$ numerically? Try the following R codes:

```
x <- seq(0.01, 1, length=100)*10^(-3)
f <- function(x){ (2*sqrt(x+1)-x-2)/(x^2) }
plot(x, f(x))
x <- seq(0.01, 1, length=100)*10^(-5)
plot(x, f(x))
```

- Definition 1. (formal definition of limit). Suppose that a and L are real numbers. $\lim_{x\to a} f(x)=L$ means that
$$\underbrace{\text{for every } \varepsilon>0}_{\text{給定誤差限制 }\varepsilon},\underbrace{\text{there exists } \delta>0}_{\text{可找到控制參數}\delta}$$
such that (for x in the domain of f)
$$\underbrace{0<|x-a|<\delta}_{\text{當}x\text{控制在}a\pm\delta\text{範圍且}x\neq a}\quad\Rightarrow\quad\underbrace{|f(x)-L|<\varepsilon}_{\text{誤差}f(x)-L\text{就可以控制在}\pm\varepsilon\text{範圍}}.$$

Example 1. Show that $\lim_{x\to 10}(2x+1)=21$.

Proof. For every $\varepsilon>0$, note that
$$\begin{aligned}&|2x+1-21|<\varepsilon\\ \Leftrightarrow\quad & x\in\left(10-\frac{\varepsilon}{2},10+\frac{\varepsilon}{2}\right).\end{aligned}$$
Take
$$\delta=\frac{\varepsilon}{2}$$
then
$$(10-\delta,10+\delta)\subset\left(10-\frac{\varepsilon}{2},10+\frac{\varepsilon}{2}\right)$$
and
$$0<|x-10|<\delta\Rightarrow x\in\left(10-\frac{\varepsilon}{2},10+\frac{\varepsilon}{2}\right)\Rightarrow|(2x+1)-21|<\varepsilon.$$
Therefore, $\lim_{x\to 10}(2x+1)=21$.

- Example 2.

  (a) Show that $\lim_{x\to 1} 2 = 2$ and $\lim_{x\to 1} x = 1$.

  (b) Suppose that k and a are two constants. Show that $\lim_{x\to a} k = k$ and $\lim_{x\to a} x = a$.

- Some observations from Definition 1.

  - **Fact 1** *In Definition 1, we may assume that $\varepsilon < 1$ (or $\varepsilon <$ some given positive number) without loss of generality.*
  - **Fact 2** $\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a} f(x) - L = 0 \Leftrightarrow \lim_{x\to a} |f(x) - L| = 0$.
  - **Fact 3** *Suppose that $a \in R$ and there exists $D > 0$ such that*

    $$f(x) = g(x) \textit{ for every } x \textit{ in } (a - D, a) \cup (a, a + D).$$

    If $\lim_{x\to a} g(x) = L$, then

    $$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L.$$

- Note that by Example 2(b) and Fact 2, $\lim_{x\to 0} |x| = 0$.

Example 3. Show that $\lim_{x\to 10} x^2 = 100$.

Proof. For every $\varepsilon \in (0, 100)$, note that

$$\begin{aligned} & |x^2 - 100| < \varepsilon \\ \Leftrightarrow\ & x \in (-\sqrt{100+\varepsilon}, -\sqrt{100-\varepsilon}) \cup (\sqrt{100-\varepsilon}, \sqrt{100+\varepsilon}). \end{aligned}$$

Take

$$\delta = \min(\sqrt{100+\varepsilon} - 10, 10 - \sqrt{100-\varepsilon}),$$

then

$$(10 - \delta, 10 + \delta) \subset (\sqrt{100-\varepsilon}, \sqrt{100+\varepsilon})$$

and

$$0 < |x - 10| < \delta \Rightarrow x \in (\sqrt{100-\varepsilon}, \sqrt{100+\varepsilon}) \Rightarrow |x^2 - 100| < \varepsilon.$$

Therefore, $\lim_{x\to 10} x^2 = 100$. Here we may assume $\varepsilon < 100$ because of Fact 1.

- Example 4. Suppose that

$$f(x) = \begin{cases} 2 & \text{if } |x-1| < 3; \\ 0 & \text{if } |x-1| \geq 3. \end{cases}$$

  Find $\lim_{x\to 1} f(x)$.

  Sol. Note that $f(x) = 2$ for $x \in (1-3, 1+3) - \{1\}$, so by Fact 3, $\lim_{x\to 1} f(x) = \lim_{x\to 1} 2 = 2$.

- Note that it follows from Fact 3 that $\lim_{x\to c} |x| = c$ for $c > 0$.

- Existence of a limit.

  - If there is a real number L such that $\lim_{x\to a} f(x) = L$, then we say that $\lim_{x\to a} f(x)$ exists. If there is no such L, then we say that the limit $\lim_{x\to a} f(x)$ does not exist.

- Basic properties for limit computation. Suppose that a is a real number. Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then the following rules hold.

  - Constant rule: $\lim_{x\to a} k = k$.
  - Limit of x rule: $\lim_{x\to a} x = a$.
  - Multiple rule: $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$.
  - Sum rule:

$$\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

  - Difference rule:

$$\lim_{x\to a} (f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$$

  - Product rule:

$$\lim_{x\to a} (f(x)g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right).$$

  - Quotient rule:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

    if $\lim_{x\to a} g(x) \neq 0$.

– Power rule for positive integer power. Suppose that m is a positive integer. Then

$$\lim_{x \to a} (f(x))^m = \left(\lim_{x \to a} f(x)\right)^m .$$

– The proofs of the above rules are based on Definition 1 except that the power rule for integer power follows from the product rule.

Example 5. Find $\lim_{x \to 1}(3x^3 - 2x^2 + x + 2)$.

Sol. $\lim_{x \to 1}(3x^3 - 2x^2 + x + 2) = 3 - 2 + 1 + 2 = 4$.

Example 6. Let $f(x) = (x - 1)/(x^2 - 3x + 2)$. Find $\lim_{x \to 1} f(x)$.

Sol. $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{x - 1}{(x - 1)(x - 2)} = \lim\limits_{x \to 1} \dfrac{1}{x - 2} = -1.$

Example 7. Let $f(x) = 1/(x^2 - 3x + 2)$. Find $\lim_{x \to 1} f(x)$.

Sol. $\lim_{x \to 1} f(x)$ does not exist. To show this, suppose that $\lim_{x \to 1} f(x)$ exists and let $L = \lim_{x \to 1} f(x)$. Then

$$\lim_{x \to 1} \left(f(x)(x^2 - 3x + 2)\right) = L \lim_{x \to 1}(x^2 - 3x + 2) = L \cdot 0 = 0,$$

which contradicts with the fact that

$$\lim_{x \to 1} \left(f(x)(x^2 - 3x + 2)\right) = \lim_{x \to 1} 1 = 1.$$

Therefore, $\lim_{x \to 1} f(x)$ does not exist.

- Squeeze rule (夾擠定理). Suppose that $a \in R$ and there exists $D > 0$ such that

$$g(x) \le f(x) \le h(x) \text{ for every } x \text{ in } (a - D, a) \cup (a, a + D).$$

Then

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) \Rightarrow \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \lim_{x \to a} h(x).$$

Example 8. Use the squeeze rule to find the following limits.

(a) $\lim_{x\to 0}\sin(x)$.

(b) $\lim_{x\to 0} x\sin(1/x)$.

Ans: (a) 0 (b) 0.

Example 9. Suppose that $n > 0$ is an integer. Show that

$$\lim_{x\to a} f(x) = 1 \Rightarrow \lim_{x\to a} f(x)^{1/n} = 1$$

using the fact that $|f(x)^{1/n} - 1| \leq |f(x) - 1|$ if $f(x) \geq 0$.

- From Example 9 and Example 8 (a), we can deduce that $\lim_{x\to 0}\cos(x) = \lim_{x\to 0}\sqrt{1-\sin^2(x)} = 1$.
- Power rule for rational power. Suppose that $\lim_{x\to a} f(x)$ exists and is positive. Suppose that r is a rational number. Then

$$\lim_{x\to a} (f(x))^r = \left(\lim_{x\to a} f(x)\right)^r.$$

Example 10. Find $\lim\limits_{x\to 2}(x^2+4)^{1/3}$.

Example 11. Find $\lim\limits_{x\to 0}\dfrac{2\sqrt{1+x}-x-2}{x^2}$.

- Change of variable. Suppose that $\lim_{x\to a} g(x) = L$ and there exists $D > 0$ such that $g(x) \neq L$ for $x \in (a-D, a+D) - \{a\}$. Then

$$\lim_{x\to a} f(g(x)) = \lim_{y\to L} f(y)$$

if $\lim_{y\to L} f(y)$ is defined.

- Example 11.1. Suppose that c is a real number. Show that $\lim_{x\to c}\sin(x) = \sin(c)$.

  Sol. Write

$$\sin(x) = \sin(c + (x-c)) = \sin(c)\cos(x-c) + \cos(c)\sin(x-c).$$

Since $\lim_{x\to c}(x-c) = 0$ and $x - c \neq 0$ for $x \in (c-D, c+D) - \{c\}$,

$$\lim_{x\to c}\cos(x-c) = \lim_{y\to 0}\cos(y) = 1 \text{ and } \lim_{x\to c}\sin(x-c) = \lim_{y\to 0}\sin(y) = 0.$$

Thus

$$\lim_{x\to c}\sin(x) = \sin(c)\cdot 1 + \cos(c)\cdot 0 = \sin(c).$$

- Example 11.2. Suppose that

$$f(x) = \begin{cases} K & \text{if } x = 2; \\ 1 & \text{if } x > 0 \text{ and } x \neq 2; \\ 0 & \text{if } x = 0; \\ -1 & \text{if } x < 0. \end{cases}$$

  and

$$g(x) = \begin{cases} 1 & \text{if } x = 0; \\ 2 & \text{if } x \neq 0. \end{cases}$$

  Then $\lim_{x\to 0} g(x) = 2$, $\lim_{x\to 0} f(g(x)) = K$ and $\lim_{y\to 2} f(y) = 1$, so for $K \neq 1$,

$$\lim_{x\to 0} f(g(x)) \neq \lim_{y\to 2} f(y).$$

- Special limits. $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ and $\displaystyle\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$.

  Example 12. Find $\displaystyle\lim_{x\to 0} \frac{\sin(0.5x)}{x}$.

  Example 13. Find $\displaystyle\lim_{x\to 0} \frac{\sin^{-1}(x)}{x}$ assuming $\lim_{x\to 0} \sin^{-1}(x) = 0$.

- One-side limits (單邊極限).
  - Definition of $\lim_{x\to a^+} f(x) = L$ is the definition given in Definition 1 with a replaced by a^+ and "$0 < |x-a| < \delta$" replaced by "$0 < x - a < \delta$".
  - Definition of $\lim_{x\to a^-} f(x) = L$ is the definition given in Definition 1 with a replaced by a^- and "$0 < |x-a| < \delta$" replaced by "$0 < a - x < \delta$".

- Suppose that Δ can be a, a^+ or a^-, then in the computation of $\lim_{x\to\Delta} f(x)$, only the x's in a "neighborhood" of Δ are relevant. We denote a relavent δ neighborhood of Δ by $N(\Delta, \delta)$:

$$N(\Delta, \delta) = \begin{cases} N(a, \delta) & = & (a-\delta, a) \cup (a, a+\delta); \\ N(a^-, \delta) & = & (a-\delta, a); \\ N(a^+, \delta) & = & (a, a+\delta). \end{cases} \qquad (1)$$

- Fact 3 can be modified as follows.

Fact 4 *Suppose that Δ can be a, a^+ or a^- and there exists $D > 0$ such that*

$$f(x) = g(x) \text{ for every } x \text{ in } N(\Delta, D).$$

If $\lim_{x\to\Delta} g(x) = L$, *then*

$$\lim_{x\to\Delta} f(x) = \lim_{x\to\Delta} g(x) = L.$$

- Example 15. Suppose that $f(x) = 1$ for $x < 1$; $f(x) = x$ for $x > 1$; $f(1) = 3$. Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.
- Suppose that Δ can be a, a^+ or a^- . Then the constant rule, multiple rule, sum/difference/product/quotient/power rules and Facts 1 and 2 still hold with a replaced by Δ.
- Squeeze rule (夾擠定理). Suppose that Δ can be a, a^+ or a^-. Suppose that $g(x) \le f(x) \le h(x)$ for every $x \in N(\Delta, D)$ for some $D > 0$, then

  $$\lim_{x\to\Delta} g(x) = \lim_{x\to\Delta} h(x) \Rightarrow \lim_{x\to\Delta} f(x) = \lim_{x\to\Delta} g(x) = \lim_{x\to\Delta} h(x).$$

  Recall that $N(\Delta, D)$ is defined in (1).
- One-sided limit theorem: suppose that a is a real number and L is a real number. Then

  $$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$$

  Example 17. Suppose that a is a real number. Show that the limit $\lim_{x\to a^+} 1/(x-a)$ does not exist.
- Change of variable. Suppose that Δ can be a, a^+ or a^-. Suppose that $\lim_{x\to\Delta} g(x) = L$.
  - Suppose that $g(x) \ne L$ for $x \in N(\Delta, D)$ for some $D > 0$, then

    $$\lim_{x\to\Delta} f(g(x)) = \lim_{y\to L} f(y)$$

    if $\lim_{y\to L} f(y)$ is defined.
  - Suppose that $g(x) > L$ for x in $N(\Delta, D)$ for some $D > 0$, then

    $$\lim_{x\to\Delta} f(g(x)) = \lim_{y\to L^+} f(y)$$

    if $\lim_{y\to L^+} f(y)$ is defined.

– Suppose that $g(x) < L$ for x in $N(\Delta, D)$ for some $D > 0$, then

$$\lim_{x \to \Delta} f(g(x)) = \lim_{y \to L^-} f(y)$$

if $\lim_{y \to L^-} f(y)$ is defined.

Example 18. Suppose that

$$f(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x = 0; \\ -1 & \text{if } x < 0. \end{cases}$$

Find $\lim_{x \to 0} f(x^2)$.

Practice problems

1. Show that $\lim_{x \to 1}(2x+3) = 5$ using the formal definition of limit.
2. Find $\lim\limits_{x \to 2} \dfrac{x-2}{x^2-x-2}$ and $\lim\limits_{x \to 2} \dfrac{x+3}{x^2-x-2}$.
3. Show that $\lim_{x \to 0} \cos(x) = 1$ using the fact that $\lim\limits_{x \to 0} \dfrac{1-\cos(x)}{x} = 0$.
4. Suppose that

$$f(x) = \begin{cases} x & \text{if } 0 < |x| < 10; \\ 1 & \text{if } x = 0; \\ 10 & \text{if } |x| \geq 10. \end{cases}$$

   Find $\lim_{x \to 0} f(x)$.
5. Suppose that

$$f(x) = \begin{cases} 1 & \text{if } x \notin \{0, 1, 2\}; \\ 5 & \text{if } x = 0; \\ 4 & \text{if } x = 1; \\ 3 & \text{if } x = 2. \end{cases}$$

   Find $\lim_{x \to 0} f(f(x))$.
6. Suppose that $f(x) = |x| \cos\left(\dfrac{1}{x^2}\right)$. Show that $\lim_{x \to 0} f(x) = 0$.
7. Find $\lim\limits_{x \to 9} \dfrac{\sqrt{x}-3}{x-9}$.

8. Suppose that

$$f(x) = \begin{cases} x & \text{if } x > 0; \\ 1 & \text{if } x = 0; \\ \cos(x) & \text{if } x < 0. \end{cases}$$

Find $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$.