

Functions and graphs

- 函數, 定義域, 值域. A function (函數) f takes an input x and maps it to a unique output, denoted by $f(x)$. The set

$$\{x : f(x) \text{ is defined} \}$$

is called the domain (定義域) of f . The set $\{f(x) : x \text{ is in the domain of } f\}$ is called the range (值域) of f .

Example 1. 圖書館查詢. Input: 關鍵字; output: 搜尋結果

Example 2. 數學函數. Input: x ; output: $1 + \sqrt{x}$. Domain: $[0, \infty)$. Range: $[1, \infty)$.

- Intervals (區間)
 - R : the set of all real numbers, which is often denoted by $(-\infty, \infty)$.
 - $[a, b]$: the set $\{x : a \leq x \leq b\}$, which is also called the closed interval (閉區間) with endpoints (端點) at a and b .
 - (a, b) : the set $\{x : a < x < b\}$. It is an open interval (開區間).
 - $(a, b]$: the set $\{x : a < x \leq b\}$
 - $[a, b)$: the set $\{x : a \leq x < b\}$
- In this course, we consider functions whose domains and ranges are subsets of R .
- 函數的相等. Two functions f and g are the same if and only if
 - (a) f and g have the same domain (denoted by D) and
 - (b) $f(x) = g(x)$ for every x in D .

Example 3. Suppose that f and g are two functions defined by $f(x) = \sqrt{x}$ for $x \geq 0$ and $g(x) = \sqrt{|x|}$ for $x \in R$. Then f and g are not the same.

- 函數的合成. The composition $f \circ g$ takes an input x and returns the output $f(g(x))$. That is, $f \circ g(x) = f(g(x))$. The domain of $f \circ g$ is

$$\{x : x \text{ is in the domain of } g \text{ and } g(x) \text{ is in the domain of } f\}.$$

Example 4. Suppose that $f(x) = \sqrt{x}$ for $x \geq 0$ and $g(x) = x - 1$ for $x \in R$. Find the domain of $f \circ g$.

Ans. $[1, \infty)$.

Example 5. Find two functions f and g such that $f(g(x)) = (x^2 + 2x + 3)^2$ for $x \in R$.

Example 6. Suppose that f and g are two functions, the domain of f is $\{2, 4, 6\}$, and the domain of g is $\{1, 2, 3, 4, 5\}$. The function values of f and g are given below:

x	$f(x)$	x	$g(x)$
2	3	1	4
4	1	2	2
6	5	3	1
		4	3
		5	2

What are the domains of $g \circ f$ and $f \circ g$? What are the ranges of $g \circ f$ and $f \circ g$?

- Inverse function (反函數).
 - For a function f , and y in the range of f , the inverse function of f maps y to the unique x in the domain of f such that $f(x) = y$.
 - The inverse of f is denoted by f^{-1} .
 - If for some y in the range of f , solving $f(x) = y$ gives more than one x values, then f^{-1} does not exist (若 f 不是一對一函數, 則 f^{-1} 不存在).

Example 7. For each f given below, determine whether f^{-1} exists. If f^{-1} exists, find it.

- $f(x) = x^3$.
- $f(x) = \sin(x)$.

Example 8. In Example 6, determine whether f^{-1} exists. If f^{-1} exists, find it.

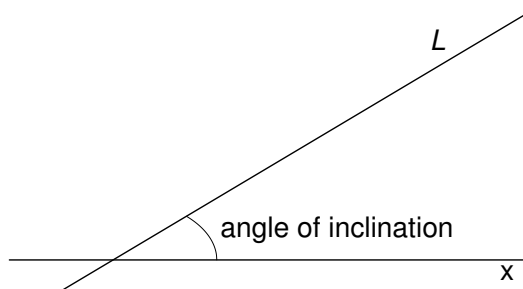
- Common functions: trigonometric functions (三角函數), polynomials (多項式), rational functions (分式型函數), exponential functions (指數函數). More functions can be obtained by taking composition, inverse, addition and multiplication of known functions.
- 反三角函數. 若將三角函數定義域適當縮小, 則反函數可定義. 例如考慮 $f(x) = \sin(x)$ for $x \in [-\pi/2, \pi/2]$ 且 f 定義域為 $[-\pi/2, \pi/2]$, 則可定義 f^{-1} . 此時 f^{-1} 記作 \sin^{-1} 或是 \arcsin .

反三角函數	\sin^{-1}	\cos^{-1}	\tan^{-1}	\cot^{-1}	\sec^{-1}	\csc^{-1}
domain	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$
range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(0, \pi)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

函數圖形相關名詞

- Angle of inclination (傾斜角, 斜角) and slope (斜率).

$$\text{slope} = \tan(\text{angle of inclination})$$



- Intercepts (截點): 函數圖形和座標軸的交點.

Example 9. $f(x) = -(x-1)(x-2)$. x -intercepts: $(1, 0)$ and $(2, 0)$.
 y -intercept: $(0, -2)$.

- Even/odd functions (偶/奇函數).
 - f is an even function if $f(-x) = f(x)$ for all $x \in R$ (偶函數圖形對稱於 y 軸).
 - f is an odd function if $f(-x) = -f(x)$ for all $x \in R$ (奇函數圖形對稱於原點).